The Shropshire
Mathematics
Experiment

Learning Mathematics

Book One

Raymond S. Heritage

Penguin Books

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The Shropshire Mathematics Experiment

1882

Learning Mathematics Book One

Raymond S. Heritage, B.Sc., F.S.S., A.F.I.M.A.

Senior Lecturer in Mathematics, Cheshire College of Education, Alsager

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To the Shropshire Mathematics Teachers, who helped to make this book

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Section 1 **Sets**

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We shall use the word 'set' a good deal in this book. We must understand clearly what it means in mathematics. If you collect stamps you will have an idea what a 'set' is in philately; you also know what is meant by a 'set' of encyclopedias. This is the right idea, but not quite clear. When you meet the word in this book, it means 'a collection' and nothing more. I can talk about the five books which are in my briefcase at present as a set, though they are: a dictionary, a mathematics book, a note-book, a detective novel and a photograph album. All that they have in common is that they are together in my bag which makes them 'the set of books in my bag now'. The members of a set always have at least one thing in common: that they all belong to the same set. Most of the sets you will meet will have more than one thing in common, or, as we say, more than one *common property*.

Exercise 1 **a** Write down three sets which can be found in your room now.

b The members of your sets have several things in common. Find three common properties for each of your sets.

c What common property is there for all the elements of all three sets?

d Could you now write down a bigger set without introducing any new members not already listed in (a)?

Exercise 2 **a** The word 'element' has a number of meanings in English. It is sometimes used in talking about the weather; it is very often used in talking about chemistry. Find out what it means in each of these and in any other uses you can find.

b All of these uses have one thing in common. What do you think it is?

c In connexion with sets, *element* means *member*. Why do you think this is?

d Here are some sets. How many elements are there in each?

i a set of pets: a cat, a dog, a budgerigar
ii a set of children: John, Susan, Mary, William, Joan
iii the set of people with red hair in your room now
iv the set of legs on a horse
v the set of corners of a triangle

e You have just given the *number of the set* in five separate examples. Can you give the number of each of the following sets? If so, say what it is and if not, explain why not.

i the set of people living in your town today
ii the set of people living in your town at the last census
iii the set of pages in this book
iv the set of grains in a bucket of sand
v the set of tunes written by German composers in the nineteenth century
vi the set of red motor cars in New Street, Birmingham on 21st September, 1965 at exactly 11 o'clock in the morning
vii the set of tunes which can be composed
viii the set of all possible numbers

f When a set has a countable number of elements, it is said to be *finite*, and when it could not under any circumstances be counted, it is said to be *infinite*.

i Look up each of these words, *finite* and *infinite*, in your dictionary.
ii Say whether each of the sets you have met in Exercise 2(d) and (e) is finite or infinite.

g Write down three examples of finite sets of which you do not know the number.

h Write down three examples of infinite sets.

i Look at Exercise 2 d (ii) and (iii) above. What can you say about the ways the sets are given?

j Write down (i) four sets by description and (ii) four more by list.

k If you do not know the number of a finite set, can you list it?

l Can you list an infinite set?

As we have already seen, we can combine sets together. Whenever we do this, the result is always itself a set. If we combine two sets, the result is called the *union* of the sets. If the sets are named A and B, we can form their union this way:

set A: a sixpenny piece and a shilling
set B: a florin and a half crown
the union of set A and set B: a sixpenny piece, a shilling, a florin, a half-crown.

Exercise 3 Say what sets are made by the unions of the following pairs of sets.

a Set A: a protractor, a ruler, a set-square
Set B: a pencil, a rubber, a pair of compasses

b Set A: Bill, Joe, Stan, George
Set B: Phil, Jack, John

c Set A: a typewriter, a packet of paper
Set B: a fountain pen

We have been writing the word *set* very often: it would be very much easier for us if we had a special sign which would save our using the word all the time. We use these: { }. They are called 'braces'. You should practise writing them for you will often need them. Instead of 'the set of books in my case', I could write: {books in my case}. The braces stand for 'the set of'.

Exercise 4 What set is the union of each of the following pairs of sets?

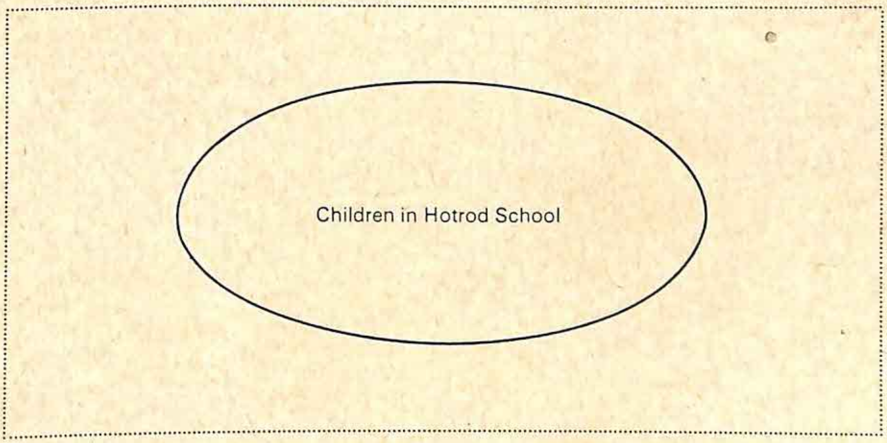
a {a fountain pen, a propelling pencil}, {a ball point pen}

b $\{a, b, c\}$, $\{d, e, f\}$

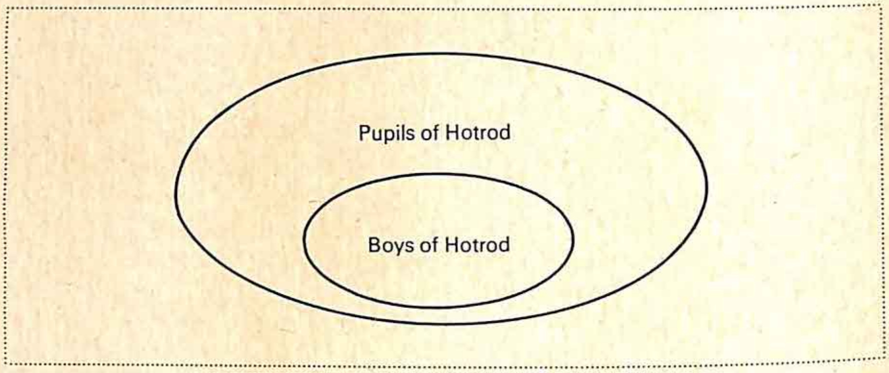
c {○, △, □}, {▭, ⬭}

d $\{X\}$, $\{Y\}$

It is sometimes useful to draw a set like this:



Now Hotrod School is a mixed school in Cornwall, so from among its pupils we can select a set of boys. This set is wholly inside the set of all the pupils of the school.



The set of boys is said to be a *subset* of the set of pupils. Here is a list of some of the possible subsets of {pupils of Hotrod}.

A = {boys}
B = {girls}
C = {pupils who play the piano}
D = {pupils who play cricket}
E = {boys who sing}
F = {boys who wear kilts every day}
G = {girls who sing}
H = {girls who wear spectacles}

Exercise 5 Look at the information above and answer the following questions.

a Can you find two subsets which overlap?

b Can you find two subsets which do not overlap?

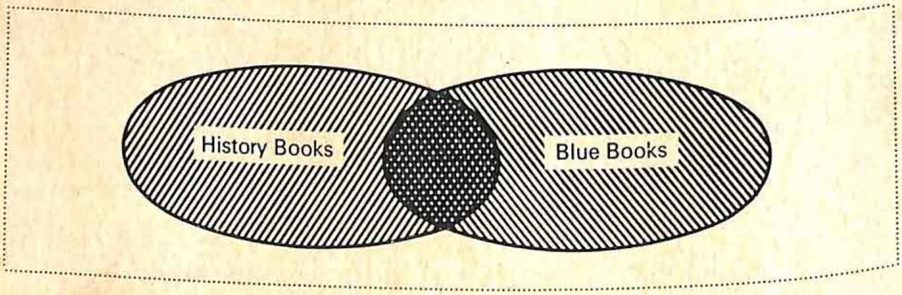
c Can you find any subsets which probably do not have any members?

d Can you find two subsets whose union is the whole set of pupils of the school?

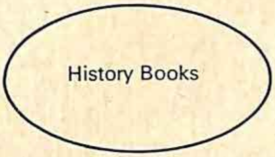
e How could you describe the set E union G (i.e. the union of sets E and G)?

If we want to draw a sketch to show two sets which overlap, we can do it like this:

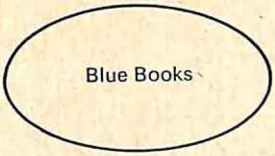
A = {all the books in my study}
B = {the history books in my study}
C = {the books bound in blue in my study}



Let us imagine the history books tied in a bundle:



and the blue books also tied in a bundle:



If any of the history books are also blue, I cannot make these two parcels at the same time, and this is the situation shown on p. 12. The union of these sets is all the history books together with all the blue books. If a book is both history and blue it cannot go in twice!

Exercise 6 Write down the sets formed by the unions of these pairs of sets:

a {5 red books}, {6 English books of which 2 are bound in red}

b {1 red rose, 1 yellow rose, 1 orange rose in the border},
{all 27 yellow flowers in the border}

c {BMC Mini-cars}, {Austin cars}

d {birds}, {farm animals}

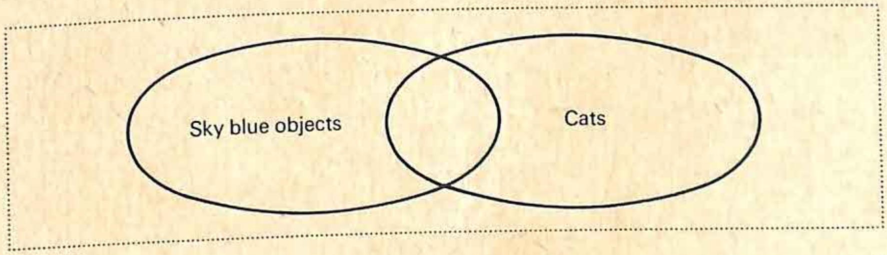
e {knaves}, {fools}

Union is equivalent to 'either ... or ...' in ordinary thinking. Look back at the questions and your answers to the previous exercise to see why that is. The last question should remind you of a well known phrase.

In the diagram of the history books and the blue books there is a part which is shaded twice – once for being blue and once for being history. This overlap is called the *intersection* of the two sets. This region illustrates those books which are *both* blue *and* history. Intersection is equivalent in ordinary thinking to 'both ... and ...'.

Exercise 7 Write down the intersections of the sets listed in Exercise 6.

Exercise 8



Look at the diagram. What is the union of the two sets mentioned?

b How would you describe the intersection of the two sets in words?

c Do you know of many examples of the intersection of the two sets? How many?

The last set you met hadn't any elements in it! We meet this set quite often; it is called the *empty set* or the *null set*.

Exercise 9 Which of the following do you think might be empty sets?

a {Dogs with two tails}

b {Cats with no tails}

c {Banana plants in your garden}

d {Spinach at the bottom of the sea}

e {tins of spotted paint}

f {Cactus plants in the Himalayas}

g {Pennies bearing the words 'Elizabeth I, regina'}

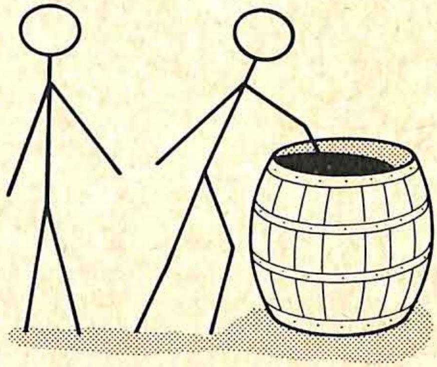
h {People with your name}

i {Immortal men}

j {Empty cigarette packets}

k {Bubbles for spirit levels}

If two sets do not overlap we sometimes say they are *disjoint sets.* What sort of set is the intersection of two disjoint sets? If you cannot decide, look back to the examples of pairs of disjoint sets in Exercise 3. When we find subsets of a set it is like putting all the elements of the set into a tub and drawing out a handful at a time. The handful is a subset. We might of course bring out a handful of nothing! This is also a subset. The empty set is a subset of every set and when we list it, it looks like this: { }.



'Have a subset'

We have signs and shorthand ways of writing some of the ideas which we have been talking about. The signs look like a code: here is the translation for you.

Words	Signs
the union of set A and set B	$A \cup B$
the intersection of set A and set B	$A \cap B$
the empty or null set	$\emptyset$
is an element of	$\in$
is contained in	$\subset$
is a subset of	$\subset$
contains	$\supset$
the number of set A	$n(A)$

Read out	when you see
A union B or A cup B	$A \cup B$
A intersection B or A cap B	$A \cap B$
phi	$\emptyset$
A is empty	$A = \emptyset$

Exercise 10 Translate the following into words:

a $A \cap C$

b $V \cup W$

c $D \subset F$

d $P \supset Q$

e $x \in H$

f $G = \emptyset$

g $n(J) = 17$

h $n(Y) = 0$

i $Y = \emptyset$

j $d \in R$

k $S \cup T$

l $X \supset Y$

m $Y \cap Z$

n $s \in S$

Exercise 11 Translate into signs:

a F is a subset of G

b union of K and J

c *d* is an element of L

d F intersection D

e H cup K

f Johnny is a member of the set of schoolboys

g M cap N

h X is contained in W

i R contains Q

j R contains the element *q*

Exercise 12 Here we shall be talking about sets of signs. You will not know the meanings of many of the signs. This will not matter for you can still do the questions.

List the sets which result from (i) the union and (ii) the intersection of the following pairs of sets.

a $\{a, b, c\}, \{b, c, d\}$

b $\{a, b, c\}, \{d, e, f\}$

c $\{\flat, \#\}, \{\flat, \natural\}$

d $\{\alpha, \beta\}, \{\gamma, \delta\}$

e $\{\mathscr{R}, \mathscr{S}\}, \{\mathscr{W}, \mathscr{X}\}$

f $\{a, b, c\}, \{a, b, c\}$

g $\{f, g\}, \{\mathrm{F}, \mathrm{G}\}$

h $\{c, a, t, s\}, \{t, s, a, c\}$

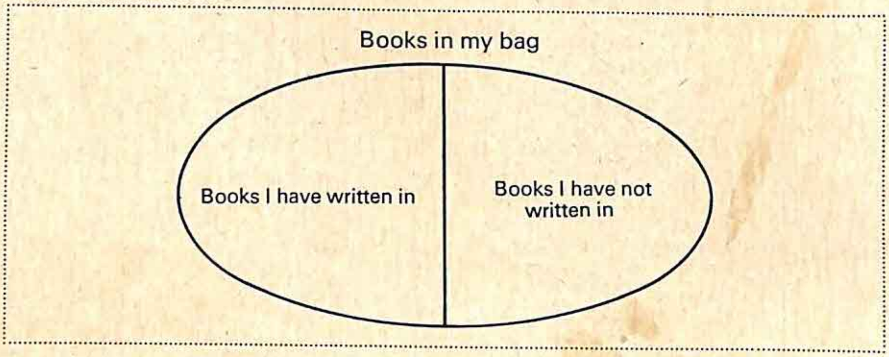
i $\{c, a, t\}, (d, o, g\}$

j $\{\mathrm{cat}\}, \{\mathrm{dog}\}$

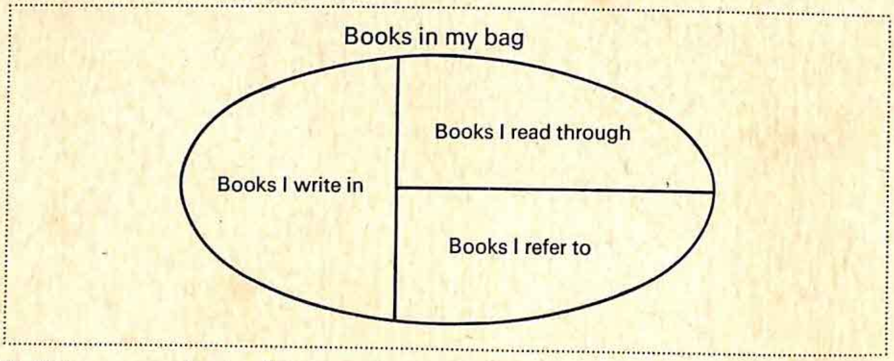
k $\{1, 2, 3\}, \{3, 4, 5, 6\}$

l $\{2, 4, 6, 8\}, \{1, 2, 3, 4\}$

Look back at the list of the set of books I said were in my briefcase (page 1). You can see that I can split them into subsets which do not overlap. Here is one way:



I could have done this a different way. For instance, I could have taken three subsets which do not overlap:



Splitting up sets in this way is called *partitioning* and the subsets are called *partitions.*

Exercise 13 **a** Partition into two sets each of the following:

i {1, 2, 3, 4, 5, 6, 7, 8, 9} (into odds and evens)
ii {Eric, Frank, Geraldine, Harry, Ivan, Joanna} (into boys and girls)
iii {trees} (one set to be 'deciduous trees')
iv {stamps} (one set 'mint')

b Partition the following into more than two sets. Make sure that the subsets do not overlap and that they do not leave anything out.

i {1, 2, 3, 4, 5, 6, 7, 8, 9}
ii {trees}
iii {cars}
iv {living creatures}
v {library books}
vi {films}

c '30 days hath September, April, June and November.
All the rest have 31 days clear,
except February alone with 28,
and 29 in each Leap Year.'

Draw a diagram to show what partition this represents

d To what sets do the following partitions apply?

i primes and composites
ii straight and curved
iii concave and convex
iv flat and curved
v convex and re-entrant
vi acute, right, obtuse, reflex

If you couldn't do these questions, try again when you have read this book right through.

Exercise 14 **a** What is a set?

b What is a subset?

c What is the word we use for the *member of a set*?

d What is meant by the *number of a set*?

e If you can count the number of members of a set, how can you describe the set?

f If there is a very large number of members in a set, too many for you to count in your life time, can you use the same word as in (e)?

g Suppose the list of members of a set goes on for ever. What word do you use to describe such a set?

h If we put two sets together to make a new set, we have formed the ... of the two sets. (Fill the gap.)

i Write down in symbols the set which contains members which are both in set A and set B.

j What two names do we give to a set with no members?

k What are the symbols for

i union
ii is a subset of
iii the number of set X
iv is a member of
v contains
vi the empty set?

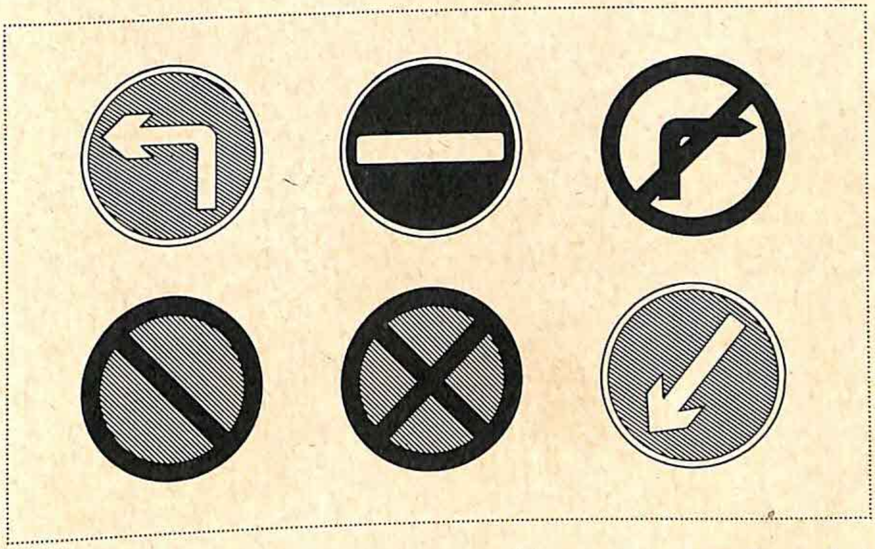
l What do these symbols mean?

i $\supset$
ii $\cap$
iii $\cup$
iv $\emptyset$

Section 2 **Symbols, counting, and calculating**

Exercise 1 **a** Somewhere near your home, you will see a shop outside which there will be a pole with a twisted candy-stripe pattern. In some towns you may find a shop with three golden balls hanging outside. What do these signs mean?

b At the roadside, you may see some of the following signs:



What do they mean? Make a list of some of the road-signs you know. Let us look carefully at these signs. Can you think of another word for the picture part of the sign?

c What is the symbol for a steep hill? What else usually appears on the 'steep-hill' sign?

d When a car is driven in a foreign country, it always bears, near the rear number-plate, a symbol indicating which country it comes from. For 'Great Britain' 'GB' is the symbol, and 'I' stands for 'Italy'. What are the symbols for the following countries: Germany, Norway, Spain, Holland? Why are these letters chosen?

e If you were to come across a scrap of paper bearing the symbols: *Slip 1 Kwys., k2, wl. fwd., kl, slip 1, k. 2 tog., p.s.s.o., k. 1, wl. fwd., k. 3.* what would you think the paper was and what would the symbols mean?

f We have now thought about four different sorts of symbols. What are symbols? Think about it very carefully before you decide what to say.

g What do the following signs or symbols stand for? @, \$, &, ?, +, =, >, <.

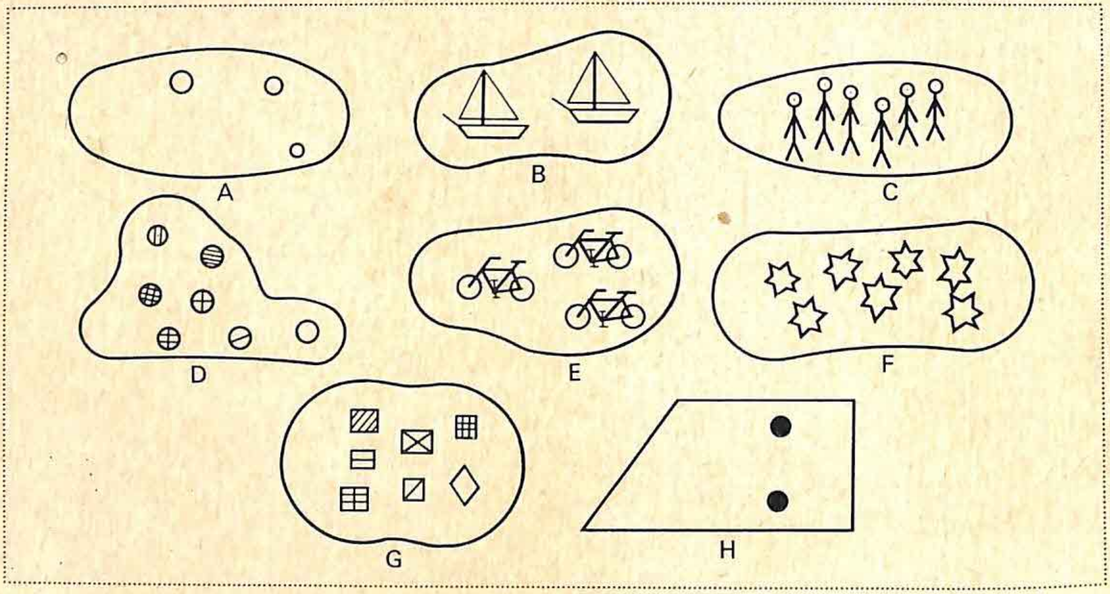
h We are very used to certain kinds of symbols – they are called 'letters' and 'words'. Letters stand for certain sounds in English. What do words stand for?

We need symbols to be able to talk at all. But it is no good saying to your neighbour: 'Pencil'. He will not know whether you want to give him one, to steal his, to ask if he has one, or to say that Bill Smith has eaten one. To say anything which he will understand, you have to make a sentence. Symbols are rarely any use by themselves. If they are used by themselves as in road signs, part of the sentence is not expressed but is 'understood'.

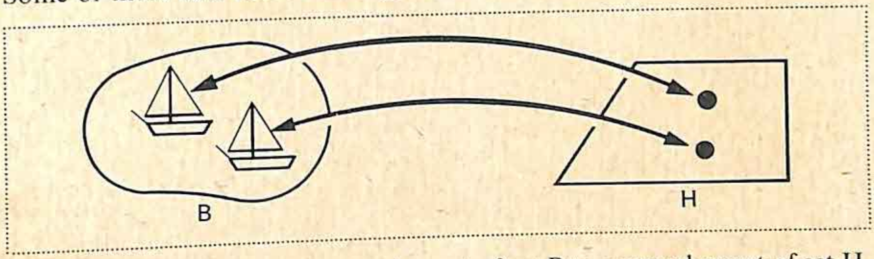
For instance this sign means 'There are road works just ahead.' The knitting pattern above, when written out in full, is entirely in sentences.

In mathematics, symbols are always used in sentences.

Here is a collection of sets:

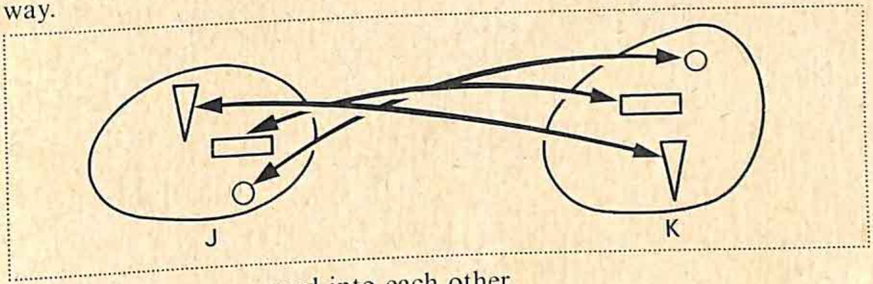


Some of these sets can be arranged like this:



There is an arrow joining every element of set B to every element of set H. There are none over in either set; no element is used twice. We have made a *one-to-one mapping*. The sets are said to be *mapped into* one another.

Exercise 2 Decide which of the sets above can be mapped into each other in this way.

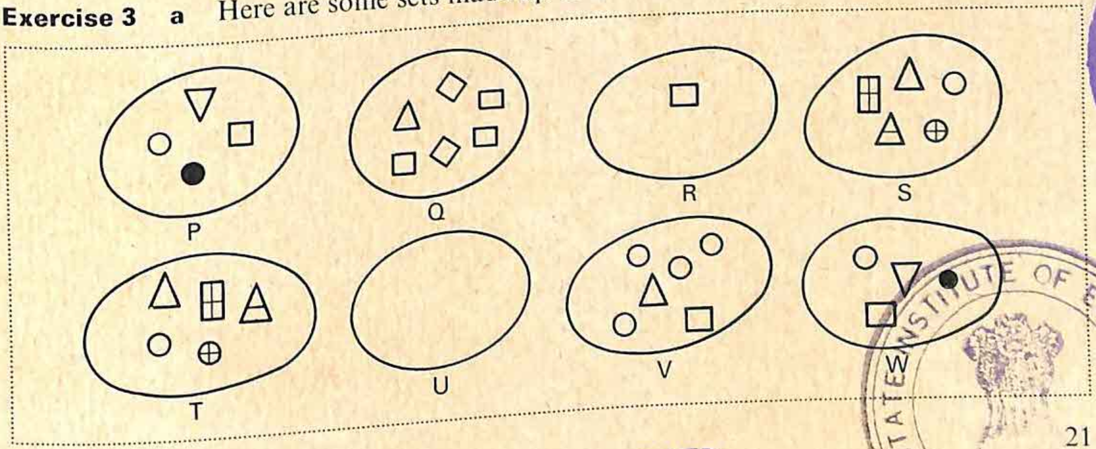


Here are two sets mapped into each other.

They can be mapped in one-to-one correspondence in several different ways. The way we have done it shows a certain fact about the sets. They are equal sets. We can write J = K.

These are also equal sets: L = $\{a, b, c, d\}$; M = $\{c, a, d, b\}$. So L = M.

Exercise 3 **a** Here are some sets made up of dots, circles, triangles and squares.



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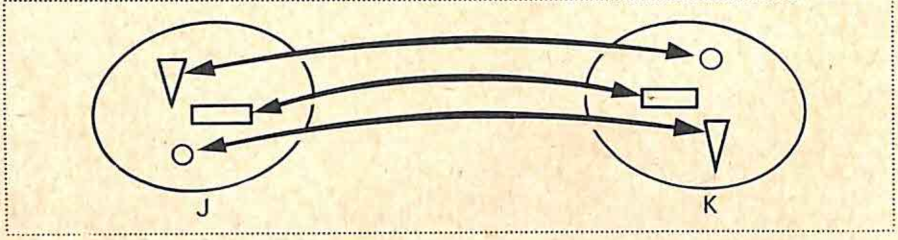
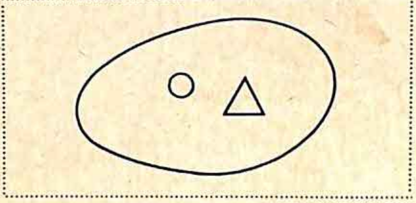
i Map them in pairs to show a one-to-one correspondence.
ii Are any of these sets equal? If so, write down statements to say so.

b Would the sets J and K still have been equal if they had been mapped like this?

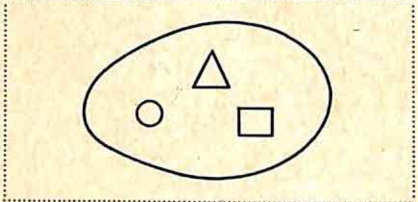


c How many different ways can you map J and K in one-to-one correspondence?

This is a set containing *two* elements. Any set which can be mapped into it in one-to-one correspondence also has two elements.

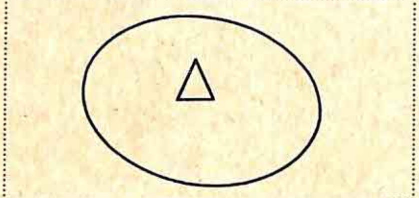


This is a set containing *three* elements. Any set which can be mapped into it in one-to-one correspondence also has three elements.

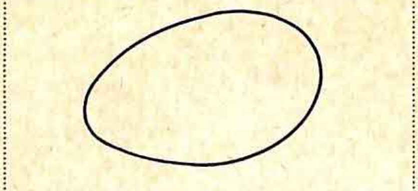


So we can go on. This is how we know what numbers are.

Any set which can be mapped into this in one-to-one correspondence has *one* element.



Any set which can be mapped into this set in one-to-one correspondence has *zero* elements.



These symbols we usually use for numbers: 0, 1, 2, 3, etc. They are called NUMERALS. You learned these long ago. Other races have not always used the symbols we use. Some of them could not write down numbers which you write very easily. The Romans for instance had no symbol for zero. Sometimes numerals were very confusing. For instance, the Hebrews used the same numeral for 1000 as for 1! There are a great many interesting facts to know about these ways of writing numbers, or, as we say, these *systems of numeration.*

Exercise 4 **a** We have drawn up a table showing the comparison of some systems of numeration. It is not finished. Make a search in suitable books for enough information to complete it. Can you also add to it an extra column or two?

Modern European	Ancient Egyptian	Ancient Greek	Roman	Hebrew
0				
1	ǀ	A	I	א
2	ǀǀ	B		ב
3		Γ		
4		Δ		ד
5		E	V	
6		F		
7		Z		
8	ǀǀǀǀ ǀǀǀǀ	H		ח
9		Θ		
10	∩	I		

When you have completed the table, find out how each of the systems is extended beyond ten.

b Find out how the Egyptian system was used to express the numbers
i 44 ii 404 iii 3003 iv 2025 v 4465

c How did the Greeks express very large numbers? Can you imagine why Greek mathematicians preferred geometry to arithmetic?

d Why did the Romans choose M and C as symbols? What did they mean?

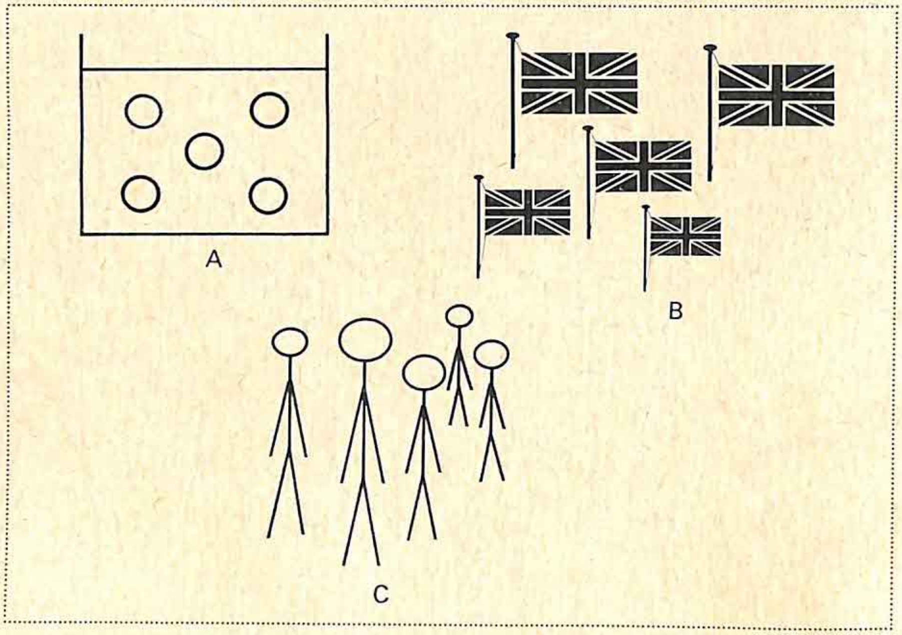
e i Put these into modern notation:
CCVI, XCIV, MDC, MCD, MCMDCCII.
ii Put these dates into Roman numerals:
1066, 1666, 1939, 1965, 2000.

Here is an exercise to remind you of some of the points we have made in this section so far.

Exercise 5 **a** We mentioned 'three balls' earlier on. The word 'three' is a symbol for a number. Is there another symbol for 'three'? What symbol do you find for 'three' on a domino? on a playing card? on a threepenny stamp?

b Now we know that '5' stands for 'five', but what does it *mean*? The five-spot of a domino, the five stones you use in the game, and the five flags, all have something in common. What is it?

c Look at the following collections of things labelled A, B, and C.



These collections are called 'sets', and each of the sets has something in common with the other two. What is it which is common?

d The number of articles in a set is called 'the number of the set'. For instance the number of the set B above is five. If we write this in symbols we could write

$$`n(B) = 5'.$$

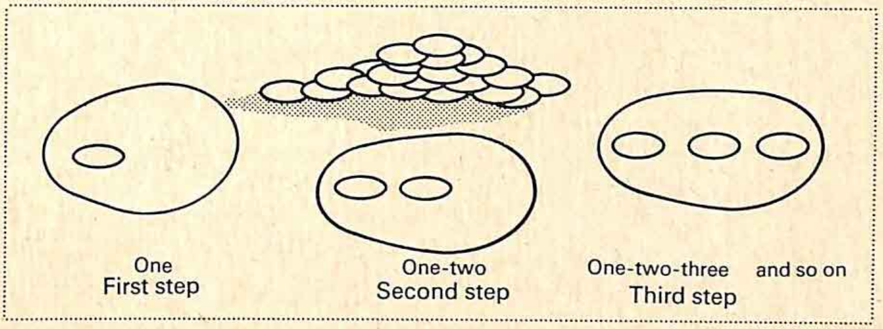
What is the number of the set of pages in this book? of the set of people in your room? of meals you ate yesterday?

e What are the numbers of the sets in Exercise 2?

f Do sets which are equal have the same number? If not, give examples.

g Are sets which have the same number always equal? If not, give examples.

The *number of a set* is sometimes called the *cardinal* number of the set. Using the word 'cardinal' allows us to use the word 'number' in another way: other than for the number of a set.
Let me count a pile of coins:



This is how we always count. We arrange the numbers in order and recite the procession of their names. We make up the set at each stage by adding one. As we add the one, we give the cardinal number of the set we have made. Each cardinal number is the successor of the one before it. If we number the steps as we have done above, or if we number the items in order inside the sets, as we have done, then we are using *ordinal* numbers. If you number the books on a library shelf: 1, 2, 3, etc., you are using ordinal numbers. The book labelled '4' is not four books, but the fourth book.

Exercise 6 **a** Is counting an operation? (That is, is it doing something?)

b Is reciting the procession of numbers the same as counting?

c Say all that you think we mean by 'seven is the successor of six'.

d Write the last statement in symbols.

e I live at number 37 in a certain street. Is '37' a cardinal or an ordinal number?

f Copy out each of the following sentences. Underline the ordinal numbers in pencil and the cardinal numbers in ink.
i Ten men and one dog went to mow a meadow.
ii On the first day of Christmas, my true love gave to me: four calling birds, three French hens, two turtle doves and a partridge in a pear tree.
iii The fifth little piggy went 'Wee, wee, wee,' all the way home.
iv Remember the fifth of November.

Sentences

When we talk, we have to use sentences.
'I red pencil' is not a sentence and is meaningless.
'I have a red pencil' is a sentence and it tells you something.
'The boy Samuel Smithers is in this class' is a sentence; so is 'The moon is made of green cheese'.
Only when we use a sentence can anybody decide whether what we say is true or false. Are the sentences above true or false?
Each of these sentences has a verb: often the verb is 'is', but it can be other things. In mathematical sentences we have verbs too; the commonest verbs are: $>$, $<$, $=$. You already know what they mean. Here are some examples, but not all are sentences. Some are true and some are false. Some are neither.

(A) $5 = 3+2$
(B) $8 < 5$
(C) $7 = 9$
(D) $6 > 4$
(E) $7 = 6-1$
(F) $1 > 9$
(G) $4+7-11$
(H) $4+7-11 = 1$
(I) $4+7-11 = 0$

Exercise 7 **a** Which of these examples are sentences?

b Which of the examples are true statements?

c Make up ten sentences using the same three verbs and say whether each is true or false.

Exercise 8 **a** How many numbers are there?

b How many numerals are there in our notation?

c Do we use special symbols for ten, eleven, twelve, etc.? Do we use the ones we have used for smaller numbers, only in a new way?

Counting larger numbers

A primitive man was once counting a pile of stones. As he picked up a stone from the pile in his right hand, he raised a finger of his left hand. This he continued to do until he had raised all the fingers and the thumb of his left hand. Here he had to stop until he realized there was a simple solution to his difficulty. When he picked up the next stone, he had to start again. So he left his first collection of stones (corresponding to the fingers and thumb of his left hand) in a small pile by themselves. This pile he called a 'hand'. So he began to count on and as he counted, he said the following numbers: 'One finger', 'two fingers', 'three fingers', 'four fingers', 'hand', 'hand and one finger', 'hand and two fingers', 'hand and three fingers', 'hand and four fingers', 'two hands', 'two hands and one finger', and so on. This way he found he could count much bigger numbers than before. What was the biggest number he could manage without inventing another new idea?

As time went on, the man realized that he could stop saying 'hand' and 'finger' each time, so instead of 'three hands and four fingers', for instance, he said 'Three, four', and this made counting much easier.

A second primitive man, much cleverer than the first, also had a pile of stones to count. He carried on the same way, except that he used the fingers and thumb of both hands. So the piles of stones which he set aside were twice as big as the other man's.

He called each of the piles a 'ten' and he counted as follows:

'One finger', 'two fingers', 'three fingers', ... 'nine fingers', 'one ten', 'one ten and one finger', and so on. What was the biggest number he could count without introducing a new idea?

He also realized that he could simplify what he said each time he moved a stone. He stopped saying 'finger' altogether, while he changed 'one ten' to 'ten'. He then changed 'two tens' to 'twenty', 'three tens' to 'thirty' and so on. He found that he could count quite easily with these new number-names.

Exercise 9 **a** If the two men each counted a set of stones, and the first man gave the number as 'four-three', what would be the number given by the second man?

b If the first man gave 'two-two', what did the second give?

c When the second man said 'twenty four' what did the first say?

d What name would the first man give to a pile of stones which the second called 'nine'?

The second man in our story counted in tens. We say that he counted *to the base ten* and this system we call the 'denary system'. The first man counted *to the base five* which we call the 'quinary system'. We could of course count to any base we like, for instance to the base eight, the octal system, or to the base two, the binary system. The base eight would have arisen if our early man had chosen to use the fingers only, and not the thumbs, of both hands, and the base two if he had used the thumbs only.

Writing larger numbers

If we want to write down our numbers using the usual symbols 1, 2, 3, 4, ... etc., we can use two numbers side by side or 'two digit numbers', e.g. 23, 31, where we are using what we call the 'place system'. (We must not read '23' as 'twenty three' unless we know that the base is ten. If it is not, we must say 'two-three'.)

To understand '23', we must know which base we are using.

23 to the base five is two fives and three.

23 to the base eight is two eights and three.

23 to the base ten is two tens and three.

Exercise 10 Check the following statements:

a 31 to the base five equals 16 to the base ten.

b 31 to the base eight equals 25 to the base ten.

c 23 to the base eight equals 34 to the base five.

Change each of the following numbers (i) to the base ten and (ii) to the base five:

d 22 to base three

e 31 to base four

f 17 to base eight

g 35 to base six

h 83 to base nine

i 32 to base nine

j Why do you have difficulty with the last but one?

k Here is a table of numbers which were worked out by James. He says that all the numbers in any row are equal. He has written the base at the top of the columns, but he has forgotten to put one above the last column. All of the numbers in that column are right but he has two wrong elsewhere in the table. Copy his table, putting in the ones he has left out, and correcting those he has wrong. What should he have written above the last column?

base eight	base ten	base four	base seven	base nine	base ?
2	2				2
			10		12
			4		4
15	14	31			23
	22			27	42
45					
	59				
	44				
	172				
			36		

Addition of numbers

The rules for addition which we know for denary numbers work quite well for other kinds of numbers providing we take care to use the 'carry-figures' properly. For instance in the quinary system, $10+4 = 14$; $4+4 = 13$. In the octal system, $10+3 = 13$; $7+7 = 16$; $10+10 = 20$. You will have noticed that some, and only some, of these calculations look precisely the same as similar denary calculations. We can make up a table of additions, or as they are sometimes called 'addition bonds'.

Here is the beginning of two addition tables

Octal

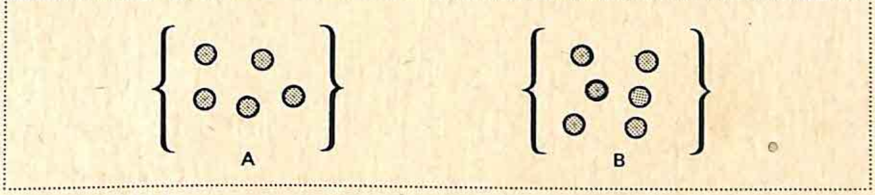
+	1	2	3	4	5	6	7	10
1	2	3	4	5	6	·	·	·
2	3	4	5	6	7	·	·	·
3	4	5	6	7	10	·	·	·
4	5	6	7	10	11	·	·	·
5	6	7	10	11	12	·	·	·
6	·	·	·	·	·	·	·	·
7	·	·	·	·	·	·	·	·
10	·	·	·	·	·	·	·	·

Denary

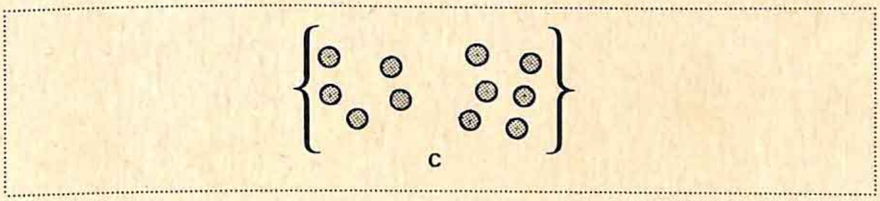
+	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	·	·	·	·	·
2	3	4	5	6	7	·	·	·	·	·
3	4	5	6	7	8	·	·	·	·	·
4	5	6	7	8	9	·	·	·	·	·
5	6	7	8	9	10	·	·	·	·	·
6	·	·	·	·	·	·	·	·	·	·
7	·	·	·	·	·	·	·	·	·	·
8	·	·	·	·	·	·	·	·	·	·
9	·	·	·	·	·	·	·	·	·	·
10	·	·	·	·	·	·	·	·	·	·

Exercise 11 Work out addition tables with ten rows and columns for the octal, denary and quinary numbers.

Exercise 12 a Look at the two sets A and B. (Regard them as piles of stones).



We can put the two sets together to form one new set.



What was the number of set A? and of set B? What is the number of the new set C? What operation in arithmetic does this represent?

b Suppose set L = {○, ⊖, ⊕}
and set M = {□, ⊟, ⊞}
and set N = {⬭, ⦶, ⊕}
and set P = {▯, ◫, ⊞}

i What is the number of each of these sets?
ii What is the number of any two of them put together?
iii What is the number of any three put together?
iv What is the number of all four put together?

c Can you express $3+3+3+3$ another way (denary system)?

d What is multiplication?

e What is 3×6 (denary system)?

f What is $6+6$ (octal system)?

g What is $14+6$ (octal system)?

h What is $6+6+6$ (octal system)?

i What is 3×6 (octal system)?

j Construct multiplication tables of 9 rows and columns in the octal system.

k Write down a multiplication table in the denary system.

Two numbers multiplied together to give a third number are said to be *factors* of the third number. A number which divides exactly into another number is a factor of the number. We can see in the denary system that a number is divisible by 10 if it ends in zero.

Exercise 13 **a** What can you discover from the denary tables about division by 2 and 5?

b What are the rules for division of octal numbers by 8?

c Can you find rules in the octal system for finding factors like those for division by factors 2 and 5 in the denary system?

d Make up multiplication tables to two other bases; one of the bases should be an odd number. Look at the tables to see what sort of patterns appear in them telling you about divisibility.

Uses of systems not based on ten

We are going to meet uses of some of these systems later on. At present we can see that shillings and pence calculations are really worked to the base twelve (the duodecimal system).

Exercise 14 Write down the bases of the following kinds of calculations:

a feet and inches

b pounds weight and ounces

c pounds sterling and shillings

d tons and hundredweights

e gallons and pints

f quarts and pints

The electronic computer

In modern life we need to perform many calculations. In factories wages, costs, and processes have to be calculated; in advanced science, very involved arithmetic is needed all the time. Among the aids which we have today there are the two sorts of electronic computers: the digital and the analogue computer. It is digital computers which people often call 'electronic brains' but they are really only very rapid calculating machines which can add and subtract at very high speed indeed. This is the type of computer we are concerned with here.

The calculations are done by using radio valves or transistors as switches, and they can be either **on** or **off**. We agree that when a valve is **on** the number **1** is represented and when it is **off**, it represents **0** (or zero). We can arrange a collection of valves to represent numbers in the binary system where we only need the numerals 1 and 0.

To understand binary numbers, we have to look at our place system again. In denary numbers 123 means *one hundred + two tens + three*. We can extend the place system as much as we like in this way

5	6	1	0	3
tens of thousands	thousands	hundreds	tens	units

$= 3\times 1+0\times 10+1\times 100+6\times 1000+5\times 10{,}000$ (reading right to left) and we simply remember that each place represents a quantity ten times as big as the one immediately to its right.

In binary numbers the places represent: ... thirty-twos, sixteens, eights, fours, twos, and units, and each place represents a quantity twice as big as that on its right. Let us compare some denary and binary numbers.

Denary	Binary
15 = one eight, one four, one two, one unit	1111
3 = one two, one unit	11
39 = one thirty two, no sixteens, no eights, one four, one two, one unit	100111
2 = one two, no units	10

Exercise 15 **a** Change the denary numbers 5, 8, 19, 63 to binaries.

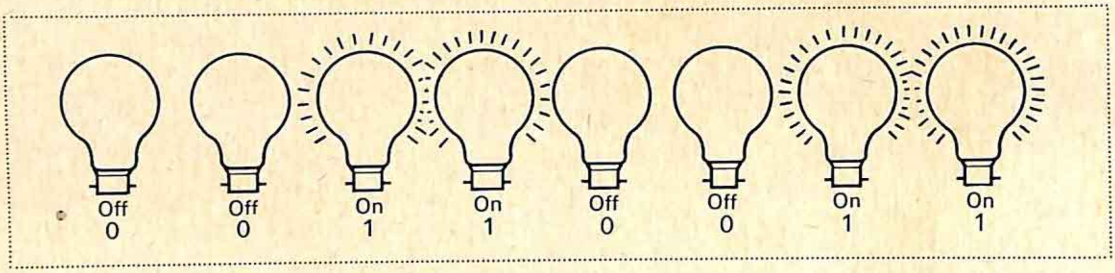
b Change the binary numbers 1100111, 11011, 101, 1101001 to denaries.

c Fill in the gaps in this table. You will discover something about binary numbers if you look carefully.

denary number	binary number	two times denary number	two times binary number
6			
9			
12			
	1111		
87			
	110111		
50			
	1110011		
128			
	10011100		
	111111111		

Addition

The addition of binaries is very simple.
$0+0 = 0, 1+1 = 10, 1+0 = 1, 10+1 = 11, 11+1 = 100$ and so on.
A digital computer is arranged so that its switches (relays, valves, or transistors) are in a row corresponding to the places of the binary numbers. The number 110011 would then be expressed thus:



If a second number is fed into the same valves, a '0' in this number does not affect the original; each '1' does however. A '1' added to a valve showing '0' (off) causes it to switch on, showing '1'. A '1' added to a valve showing '1' (on) causes it to switch off, showing '0' *and at the same time* adds a '1' to the next valve to the left.

Exercise 16 Show how the valves illustrated above will look when the number 00100101 is added to the number already shown.

Computers are built which will carry out thousands of operations in one second. These high speeds are useful in reducing the amount of labour in doing long calculations. They are essential when we want to steer space vehicles, for this is the only way the arithmetic needed can be done quickly enough.

Exercise 17 Complete the two tables below for binary numbers:

+	0	1
0		
1		

×	0	1
0		
1		

There are no other tables needed!

Read these examples carefully and be sure that you understand them. The numbers in brackets are the denary equivalents. If you have difficulty, refer to the addition tables.

$$\begin{array}{r} 101\ (5)\\ +\ 10\ (2)\\ \hline 111\ (7)\end{array}\qquad \begin{array}{r} 101\ (5)\\ +\ 100\ (4)\\ \hline 1001\ (9)\end{array}\qquad \begin{array}{r} 10\ (2)\\ +\ 11\ (3)\\ +110\ (6)\\ +101\ (5)\\ \hline 10000\ (16)\end{array}\qquad \begin{array}{r} 10111010\ (186)\\ 11011100\ (220)\\ \hline 110010110\ (406)\end{array}$$

Exercise 18 Work out these additions in binary numbers. Check your work by changing them all to denaries and adding again.

a 11+10

b 10+11

c 111+100

d 111+11

e 11+110

f 110+111

g 1010+11

h 10+1011

i 110+1011

j 1101+111

k 1011+111

l 1111+11

m 100111+11

n 1111+10

o 10+100111

p 10011+1111

q add up the numbers in the table above Exercise 15

Exercise 19

a 11+10+111

b 11+11+10

c 11+10+10

d 111+100+11

e 11+110

f 10+11+110

g 111+11+11

h 1010+1011+1101

i 1111+1111+1111

j 11+11+11+11

Numerals, digits and bits

Men probably counted on their fingers when they first invented counting. This may be the reason that many races have used base ten for their arithmetic.

We often call the separate parts of a number *digits*. For instance 457 contains three digits, a four, a five and a seven. There are two digits in 66 and seven in 1,000,000. Why do you think the word 'digit' is used?

(Look up the word in a good dictionary; this may help you.)
When we write numbers in the binary system, there are only two different numerals: 1 and 0. In the binary system they are called *bits*. 10110111 contains eight bits. Why do you think the word 'bits' was chosen? (Sorry, your dictionary will not help this time!)

Multiplication You have noticed before that $5+5+5 = 3\times 5$. In the same way $101+101+101 = 11\times 101$. You can use this to work out multiplications of binary numbers provided thay are not too big.

Exercise 20 Use this idea to work out

a 110×11 **b** 101×111 **c** 1101×111

When we are multiplying we work with one bit at a time; the result of each step being a 'partial product'.
So 1011×101 is done as follows:

$$\begin{array}{lll}
 & & 1011\ (11) \\
 & & \times\ 101\ (5) \\ \cline{3-3}
\text{i} & \text{partial product by 1} & 1011 \\
\text{ii} & \text{partial product by 0} & 0000 \\
\text{iii} & \text{partial product by 1} & 1011 \\ \cline{3-3}
 & \text{result} & 110111\ (55)
\end{array}$$

One very important question: Why do we position the partial products in this way? Be sure that you thoroughly understand why. Compare this with the multiplication of denary numbers and the positioning of the partial products. Is there any reason why we should put the partial products in that order (a, b, c,) or could we put them in a different order (e.g. c, b, a or a, c, b)? Could we leave any of them out altogether? When can we leave out a partial product completely?

Exercise 21 Work out the following multiplications in the binary system:

a $\begin{array}{r}1011\\ \times\ 11\end{array}$ **b** $\begin{array}{r}1101\\ \times 1101\end{array}$ **c** $\begin{array}{r}1000\\ \times\ 101\end{array}$ **d** $\begin{array}{r}1101\\ \times 1010\end{array}$

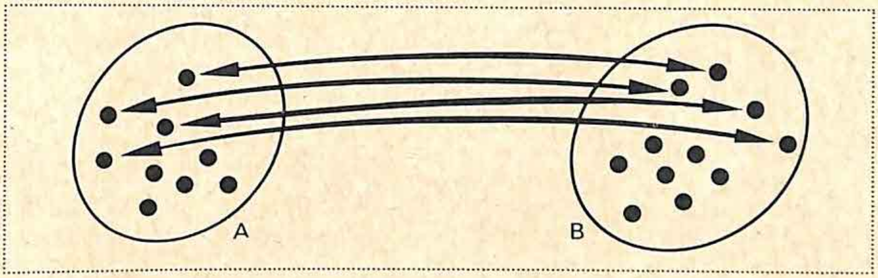
Work out the following multiplications in the denary system arranging partial products in various different ways.

e $\begin{array}{r}52\\ \times 31\end{array}$ **g** $\begin{array}{r}80\\ \times 97\end{array}$ **i** $\begin{array}{r}59\\ \times 483\end{array}$ **k** $\begin{array}{r}517\\ \times 612\end{array}$ **m** $\begin{array}{r}3571\\ \times\ 926\end{array}$

f $\begin{array}{r}64\\ \times 57\end{array}$ **h** $\begin{array}{r}8460\\ \times\ 600\end{array}$ **j** $\begin{array}{r}976\\ \times\ 78\end{array}$ **l** $\begin{array}{r}7080\\ \times\ 605\end{array}$ **n** $\begin{array}{r}7531\\ \times\ 629\end{array}$

Section 3 **More about calculating**

Exercise 1 Here are two sets. The diagram shows a mapping in one to one correspondence. It is not complete. Copy the diagram and go on with the mapping as far as you can.



a What can you say about the numbers of the two sets?

b Say which of these statements is correct:

i $n(A) > n(B)$
ii $n(A) = n(B)$
iii $n(A) < n(B)$

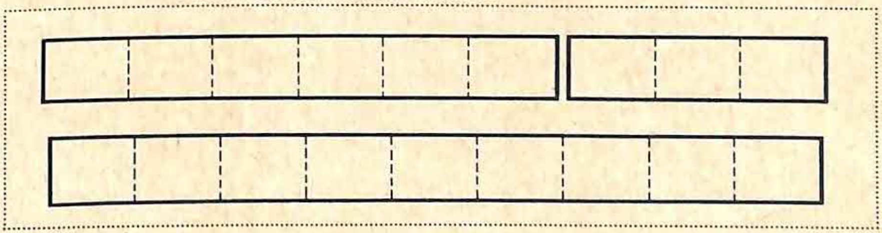
c How many elements of B remain unused in the mapping?

d Write down three statements using the symbols 2, $n(A)$, $n(B)$ together with an = sign and either a + or a − sign. All of your statements must be true ones.

e When we read out a − sign, what word do we use?

f How many ways can you combine the numbers 6, 2, 4 with an = sign and either a + or a − sign to make true statements?

Exercise 2



These three rectangles represent rods whose lengths stand for numbers.

By taking the rods two at a time we can represent each of the following statements:

$6+3 = 9$; $9-3 = 6$; $9-6 = 3$

Make sketches to show how this can be done.

How can we work out subtraction? There are many ways; none of us uses the same way all the time. Look at $27-19$. Let us find as many ways of doing it as we can.

I We can recite numbers in order starting at 19 and finishing with 27. Each time we say a number after 19 we bend down a finger. When we get to 27 we have bent down 8 fingers. This is the answer. In this way, men who knew no arithmetic could do subtraction. They only needed to be able to count. Have you ever seen a small child doing this? It is *counting-on.*

II If you buy some sweets costing 1s. 3d. and give the shop keeper 2s. he does something very like counting on. He gives you 3d. and says '1s. 6d.' Then he gives you 6d. and says '2s.' This is called *complementary addition.* We can do this with 27–19. Start with the 19, add 1, then add 7. How many have you added? This is again the answer.

Exercise 3 Work out the following by complementary addition.

a
i $56-48$ iii $1678-999$ v $236-80$
ii $37-18$ iv $125-75$ vi $101-20$

b
i 7s. 6d. − 5s. 11d. iii 10s. − 1s. 2d.
ii 5s. − 2s. 8d. iv 15s. − 6s. $4\frac{1}{2}$d.

Exercise 4 **a** What can you say about the answers to the following?
i $17-9$ ii $27-19$ iii $37-29$ iv $12347-12339$

b When I made up those questions I began with $17-9$ and then I added something to each number, and then something else and so on. Make up a series of five subtractions for yourself the same way, starting with $19-5$.

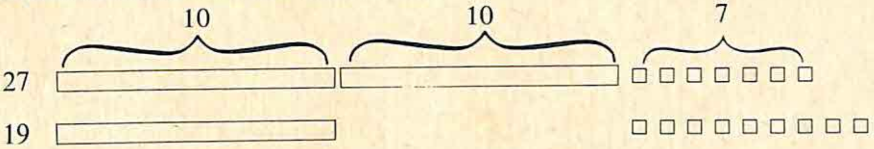
c Up to now, we have been adding so many tens to our two numbers. Does it still work if you add other numbers? Try 5, and then 7 and so on.

We can now look at method III for subtracting.

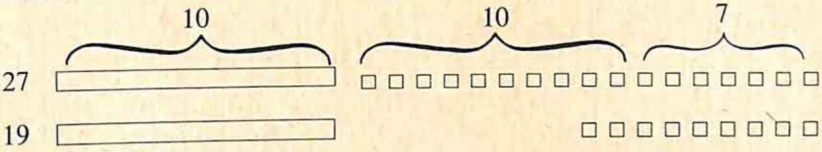
III We can see the answer to $17-9$ straight away, or we can work it out by complementary addition. When we want to work out $27-19$ we can use the result of $17-9$ this way:

$$\left.\begin{aligned} 27 &= 10+17 \\ 19 &= 10+\ \ 9 \end{aligned}\right\} \text{ so we take the 10 from the 10 and the 9 from the 17.}$$

10 Children sometimes use rods for this:



One of the 10-rods is then changed for 10 units:



We can put this down in symbols this way:

$$\begin{array}{r} {}^{1}\cancel{2}\;{}^{1}7 \\ -\;1\;\;9 \\ \hline 8 \end{array} \quad \text{or another example:} \quad \begin{array}{r} {}^{4}\cancel{5}\;{}^{1}6 \\ -\;3\;\;8 \\ \hline 1\;\;8 \end{array}$$

We have *decomposed* the 27, so this is called the *decomposition* method.

Exercise 5 Do the following by decomposition:

a i $26-8$ ii $52-25$ iii $61-38$ iv $121-17$ v $101-29$ vi $127-19$

b i $236-49$ ii $129-58$ iii $3461-2573$ iv $8973-8888$ v $1641-987$ vi $12345-6789$

IV There is one other important method for subtraction. If we want to work out $27-19$ we do $37-29$ instead! This is how it works:

we put down $\begin{array}{r} 2\;{}^{1}7 \\ -\;{}^{2}\cancel{1}\;\;9 \\ \hline 8 \end{array}$

and say 9 from 17, 2 from 2'. This is more often written down as $\begin{array}{r} 2\;\;{}^{1}7 \\ -\,1_{1}\;\;9 \\ \hline 8 \end{array}$

This method is called the method of *equal additions*. We added 10 to the 27 and then 10 to the 19.

Exercise 6 Use the method of equal additions to work out the questions of Exercise 5.

Exercise 7 Look at the following and explain why each is true.

a $127-99 = 27+1$

b $83-58 = 23+2$

c $1000-80 = 900+20$

d $1000-77 = 900+20+3$

e $996-77 = 923-4$

Exercise 8 **a** Use the shortest methods you can find to do the following and then do each again by another method.

i $56-39$ ii $127-98$ iii $256-184$ iv $5005-979$

b Explain why you can check a subtraction this way:

$$\begin{array}{r}54\\-38\\\hline 16\end{array}\qquad\begin{array}{r}38\\+16\\\hline 54\end{array}$$

In the binary system, subtraction is very simple. There can only be the following steps to remember:

$1-1=0;\quad 1-0=1;\quad 10-0=10;\quad 10-1=1$

Check these by the method of Exercise 8 b. Be sure that you understand them, especially the last.

To do longer calculations we must remember we are working in the binary system but otherwise carry on as usual. Check these calculations: there is a deliberate mistake in one of them.

binary	denary	binary	denary
$\begin{array}{r}110\\-\ 11\\\hline 11\end{array}$	$\begin{array}{r}6\\-3\\\hline 3\end{array}$	$\begin{array}{r}1101010\\-\ 101101\\\hline 111101\end{array}$	$\begin{array}{r}106\\-\ 45\\\hline 61\end{array}$
$\begin{array}{r}100\\-\ 1\\\hline 11\end{array}$	$\begin{array}{r}4\\-1\\\hline 3\end{array}$	$\begin{array}{r}1111010\\-\ 1101\\\hline 1110101\end{array}$	$\begin{array}{r}122\\-\ 13\\\hline 109\end{array}$

Exercise 9 Work out the following in binary numbers. Check each one by converting the question to denary numbers and working it out again.

a $110110-101$

b $101111-1010$

c $100001-111$

d $1111111-10101$

e $10101010-10011$

f $110110110-1010101$

Exercise 10 **a** Work out each of the following subtractions. (They are in the denary system.)

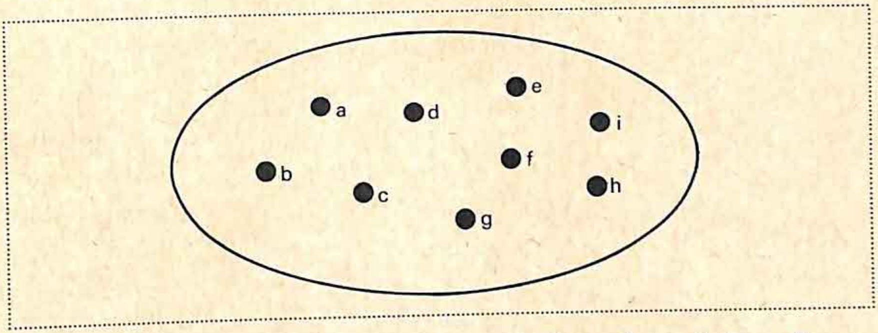
i $16-4-4-4-4$

ii $25-5-5-5-5-5$

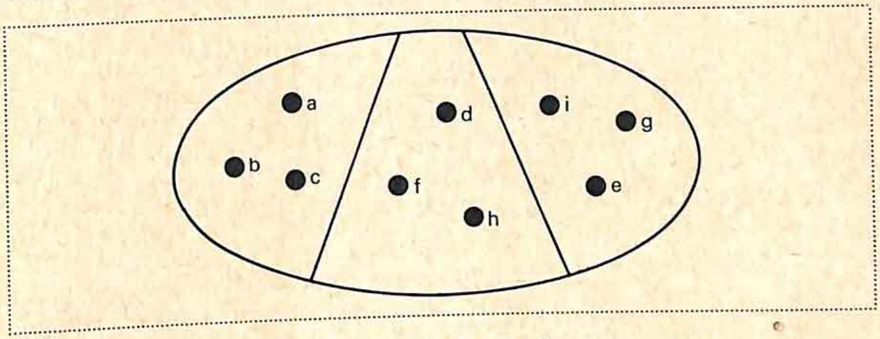
iii $18-6-6-6$

iv $12-4-4-4$

b Here is a set. It contains nine elements.



The set can be partitioned into subsets with three elements in each.



Explain what this has to do with $9-3-3-3$.

c i How many times can you subtract 25 from 100?

ii How many times can you subtract 14 from 98?

iii How many times can you subtract 17 from 85?

iv How many times can you subtract 29 from 232?

d i How many times can you subtract 25 from 108 leaving a remainder less than 25?

ii How many times can you subtract 14 from 106 leaving a remainder less than 14?

iii How many times can you subtract 13 from 87 leaving a remainder less than 13?

iv How many times can you subtract 9 from 150 leaving a remainder less than 9?

e What is the process you have really been doing?

Exercise 11 **a** i What number multiplied by 5 gives 35?
ii What number multiplied by 35 gives 105?
iii What number multiplied by 27 gives 135?
iv What number multiplied by 15 gives 90?

b i How many times can you subtract 5 from 35?
ii How many times can you subtract 35 from 105?
iii How many times can you subtract 27 from 135?
iv How may times can you subtract 15 from 90?

c Explain what the answers to (b) have to do with the answers to (a).

d i How many times can you subtract 7 from 35?
ii Make up questions to correspond to ii iii and iv in **a** and **b** above.

e

$$6 \times 7 = 42; \qquad 42 \div 6 = 7; \qquad 42 \div 7 = 6$$

These three statements are different ways of saying the same thing. Make up three of your own from each of the following sets of numbers.

i 5, 3, 15.
ii 7, 3, 21.
iii 9, 7, 63.
iv 8, 5, 40.
v 8, 17, 136.
vi 4, 27, 108.

We can now tackle division in the binary system. We do it exactly as we would with denary numbers. Divide 1111 by 101. In symbols: $1111 \div 101$ the usual layout is $101\overline{)1111}$ and the answer is, step by step, placed above the line. We are dividing by a three bit number so we shall begin by taking the first three bits of the number to be divided. The first three, 111, form a number greater than 101, the dividing number. $(111 > 101)$. 101 goes into 111 once. Carry out the first stage:

$$\begin{array}{r} 1 \\ 101\overline{)1111} \\ \underline{101} \\ 10 \end{array}$$

Bring down the next figure, 1, and divide again.

$$\begin{array}{r} 11 \\ 101\overline{)1111} \\ \underline{101} \\ 101 \\ \underline{101} \\ 000 \end{array}$$

The result is 11.

Now follow through these worked examples to find out if there are any mistakes in them.

$$\begin{array}{r} 101 \\ \text{A } 110\overline{)11110} \\ \underline{110} \\ 11 \\ \underline{00} \\ 110 \\ \underline{110} \\ 000 \end{array} \qquad \begin{array}{r} 11 \\ \text{B } 110\overline{)10010} \\ \underline{110} \\ 110 \\ \underline{110} \\ 000 \end{array} \qquad \begin{array}{r} 11 \\ \text{C } 1100\overline{)100101} \\ \underline{1100} \\ 1101 \\ \underline{1100} \\ 1 \end{array}$$

In C there is 1 left when we have done as much division as we can. This is a remainder. When we divide natural numbers there is often a remainder. Later you will find a sort of arithmetic in which the remainders are the important part. Remember that if you are working with natural numbers only, you cannot have fractions in your answers. There are no fractions in the natural number system.

Compare the process you have done above with the following denary process.

$$\begin{array}{r} 255 \\ 25\overline{)6378} \\ \underline{50} \\ 137 \\ \underline{125} \\ 128 \\ \underline{125} \\ 3 \end{array}$$

Exercise 12 **a** Which division process do you find easier, binary or denary? Why?

b Can you simplify the layout of the worked example A? (We leave out lines of certain kinds as a rule.)

c Convert all four of the worked examples of binary division to denary calculations to check them.

d Now do these binary divisions yourself.

i $110 \div 11$
ii $1101 \div 11$
iii $1011 \div 11$
iv $11110 \div 101$
v $11111 \div 110$
vi $10011 \div 111$

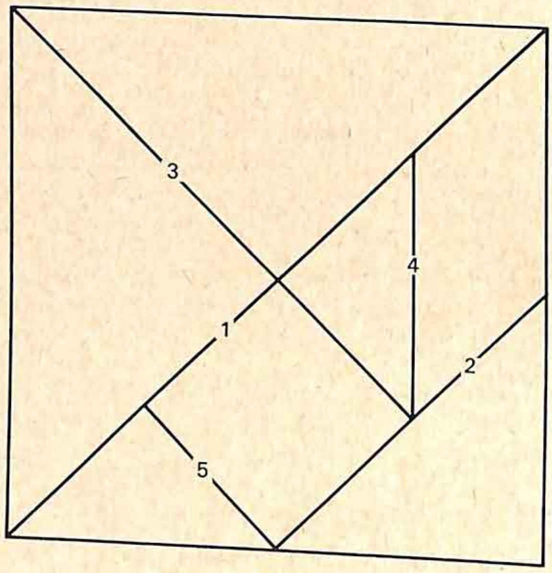
e Now do these denary divisions.

i $413 \div 7$
ii $4956 \div 14$
iii $214{,}052 \div 59$
iv $3877 \div 68$
v $249{,}968 \div 385$
vi $507 \div 36$

Section 4 **Drawing**

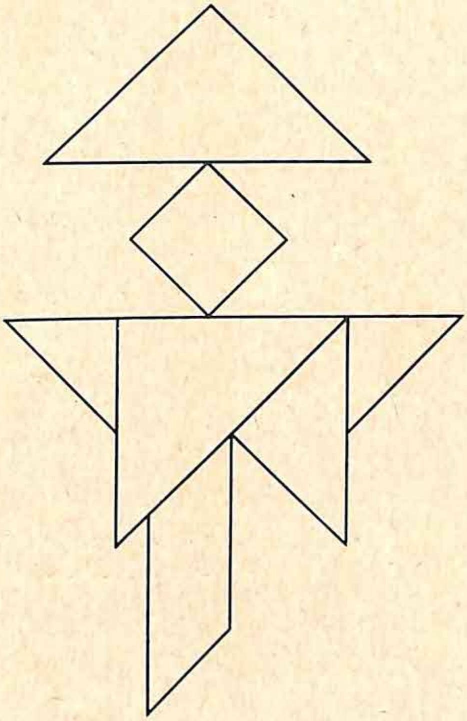
For the work of this section you are going to need a ruler, a sharp pencil, a set square, a pair of scissors, four pieces of thin card about 6 in. by 6 in., probably a rubber and certainly a lot of care.

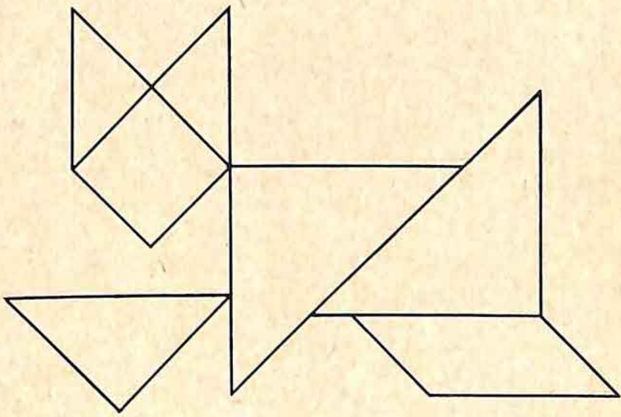
Exercise 1 Draw a square with its sides 4 in. long. Copy on to it lines corresponding to the firm lines in the diagram. We have numbered the firm lines to help you do them in order without making mistakes.



a Cut out the square and then dissect it by cutting along the lines you have drawn. You should have five right angled triangles (two pairs and an odd one), one square and a parallelogram. Which is which? Shapes which fit each other exactly are said to be *congruent*. Have you any congruent figures?

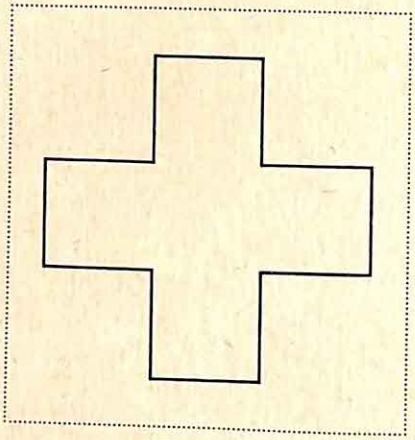
b Rearrange your shapes to make the following shapes.





c Can you find any other interesting shapes to make?

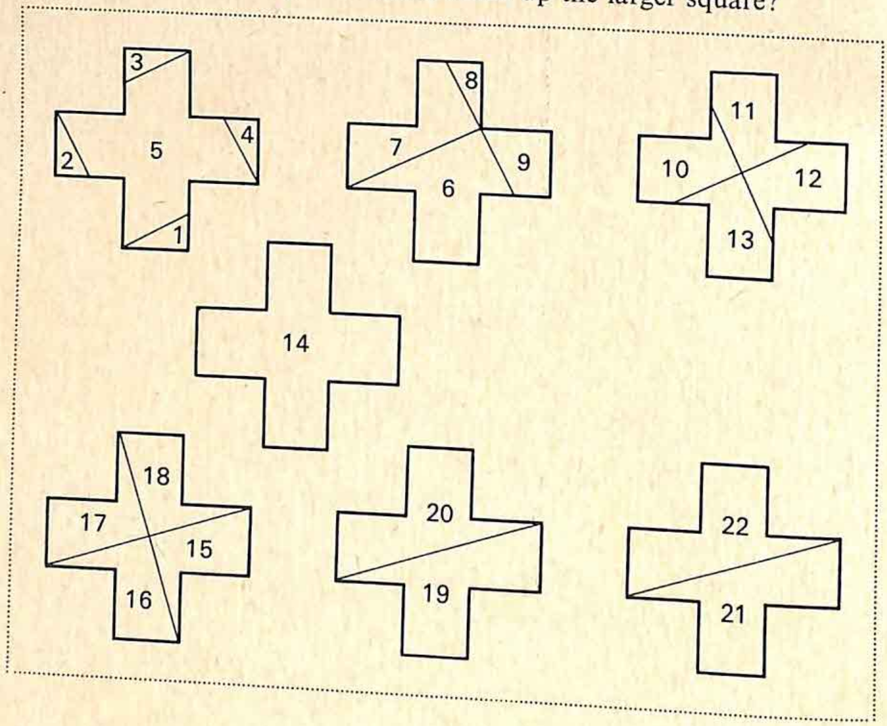
Exercise 2 The cross we have drawn is made up of 1 in. squares.



a How many squares are there to each cross?

b Could you start from a large square and remove smaller squares to get the same figure? If so:

i What would be the size of the large square?
ii What would be the size of the smaller squares?
iii How many smaller squares would you be removing?
iv How many small squares would make up the larger square?



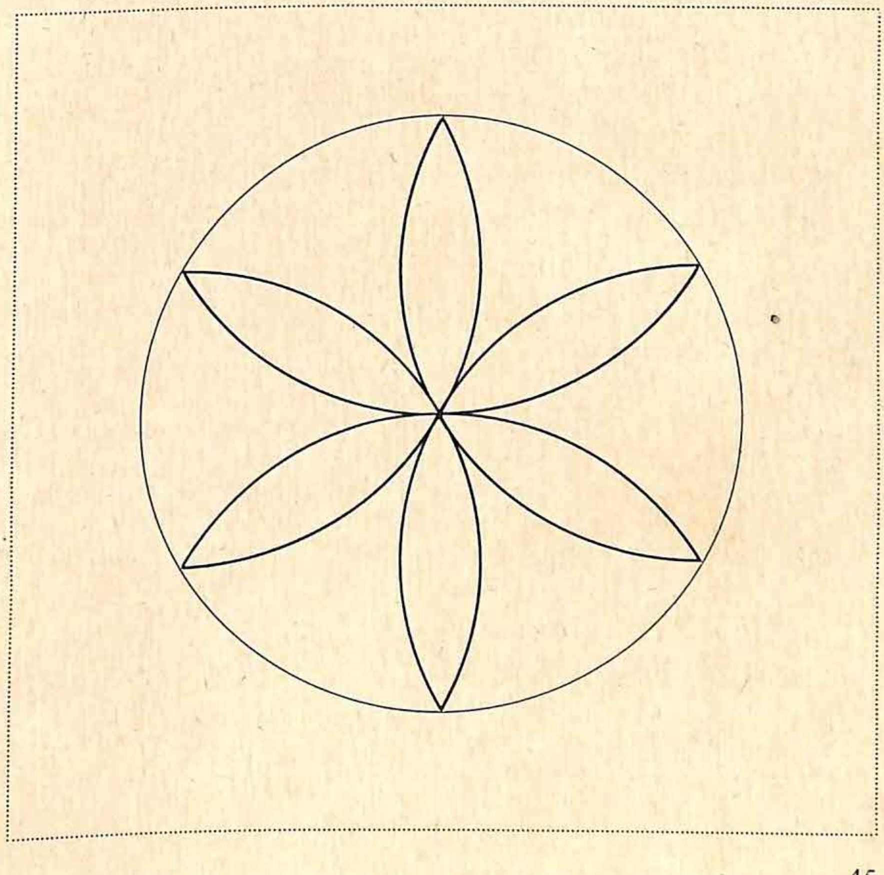
c On your card draw seven of these crosses. There is no need to make each cross of five 1 in. squares; any size will do, but make all seven the same size. Dissect them as shown and number them before you cut them out.

As soon as you have cut them out number each one on the reverse side too.

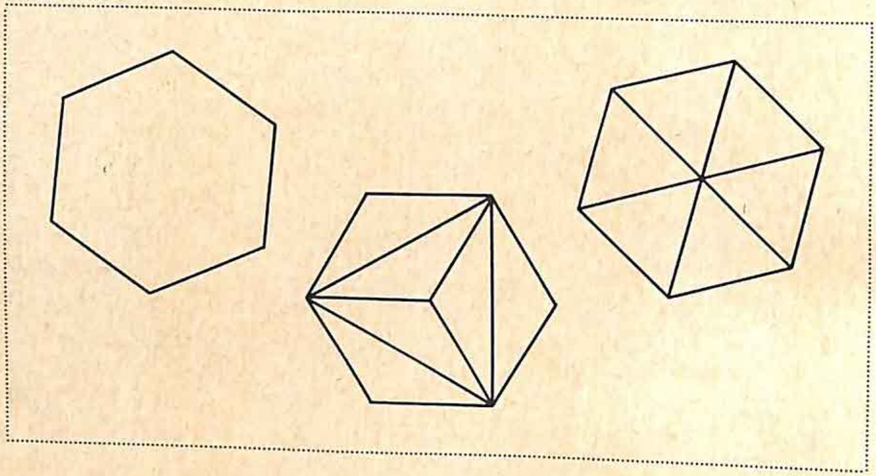
d i Rearrange pieces 1, 2, 3, 4, 5 to form a square.
  ii Rearrange pieces 10, 11, 12, 13 to form a square.
  iii Do pieces 6, 7, 8, 9 form a square?
  iv Put 14, 15, 16, 17, 18 together to form a square.
  v Do 19, 20, 21, 22 also form a square?

Exercise 3 For this exercise you will need your pair of compasses.

a Copy this diagram on paper. It should be perfectly symmetrical when you have finished. All the lines should cross at the centre, and all the petals should have sharp points on the circle.



b Begin as before and step off the six points round the circle. Do this three times with circles the same size on a piece of your card. Join the points as shown in our three diagrams. The drawing on the left is six-sided and is called a *hexagon*. It has all its sides equal, so it is said to be a *regular* hexagon. Cut out and dissect the figures as shown. Can you rearrange all thirteen pieces to form a larger hexagon?



Exercise 4

This exercise is in fact a famous old puzzle. In Exercise 1 you drew a cross made up of five squares. Needless to say the five squares can themselves be rearranged in many different ways.

a How many ways? Cut out five squares and try it, but do not forget to note each one as you discover it.

b Why do you think some people call these shapes *pentominoes*?

c Now carefully draw all the possible pentominoes onto a piece of card or graph paper glued down to card and cut them out. Can you rearrange these pieces to form a rectangle 10 units long by 6 units wide? (The *unit* referred to is the length on the side of the small square you began with.) You should be able to do this for there are 2,339 ways and they are all different!

d Cut out now a *tetromino*, that is an arrangement of four small squares (the same size as before.) The tetromino you will need will be a square. Add this to your pentominoes and see if you can now make an 8 unit by 8 unit square. There are sixty-five ways of doing this with your tetromino in the middle.

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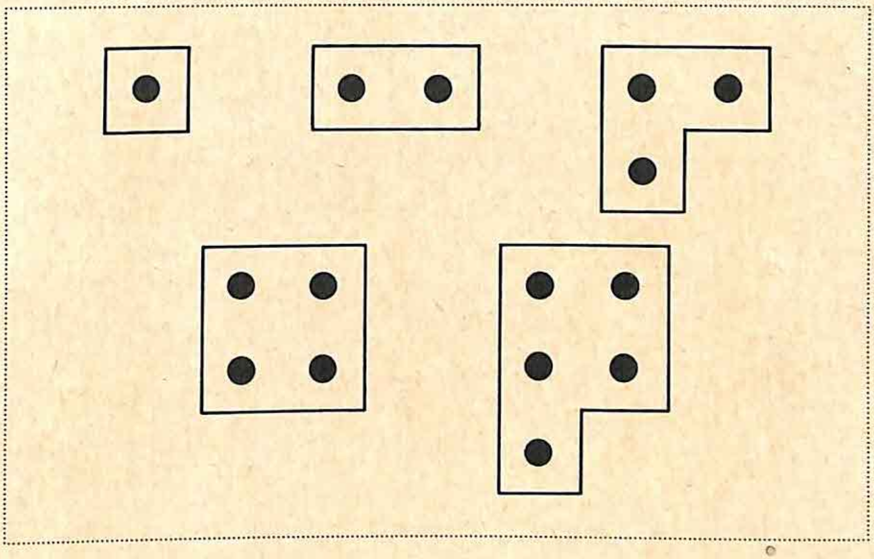
Section 5 **Odds, evens and primes**

Exercise 1 **a** Draw a set of patterns with dots to illustrate the numbers on

i dominoes ii playing cards.

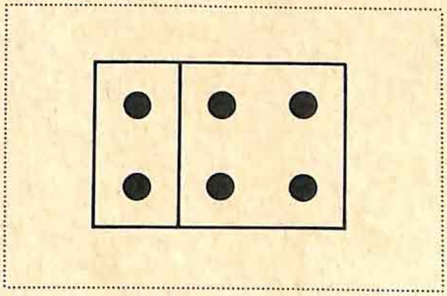
b Continue the following pattern up to ten:



c How can you recognize an even number?

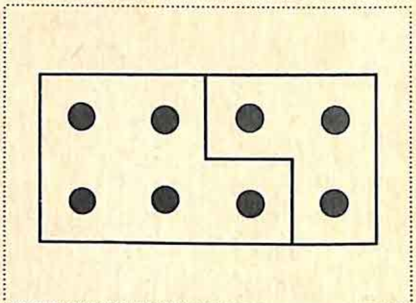
d How can you recognize an odd number?

e If you place the two-pattern and the four-pattern together you get the six-pattern. Are two, four and six all even?



f If you add together two even numbers, what kind of number do you get? Does it *always* happen? Give your reasons.

g Place the five-pattern with the three-pattern. What sort of number do you get?



h If you add together two odd numbers, what kind of number do you get?

i What kind of number results from adding an odd and an even?

j Is the sum of three even numbers, even or odd?

k Is the sum even + even + even + even, even or odd?

l If an even number is added to itself an even number of times, is the result even or odd?

m If an even number is added to itself an odd number of times, is the result even or odd?

n An even number of odd numbers is added up. Is the result even?

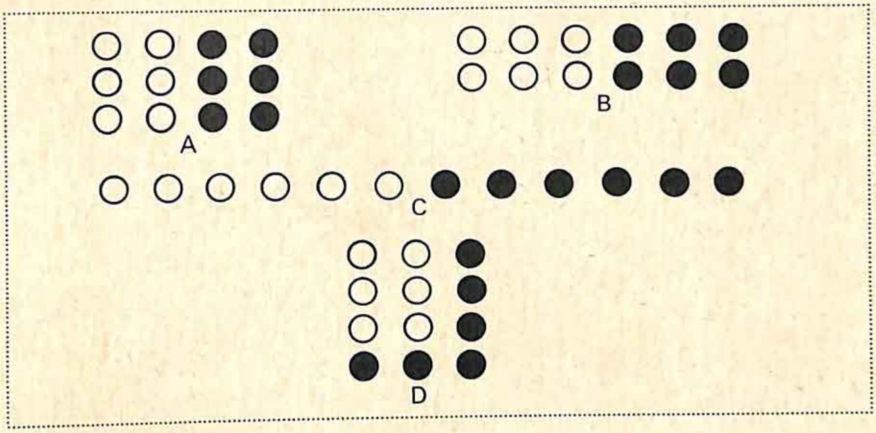
o An odd number of odd numbers is added. Is the result even this time?

p The evens and odds go alternately along the counting sequence of natural numbers. Is zero even or odd?

q Complete the following table:

	even or odd
i even × 2	
ii even × 3	
iii even × 4	
iv even × 5	
v even × even	
vi even × odd	
vii even × natural number (not 0)	
viii even × zero	

	even or odd
ix odd × 2	
x odd × 3	
xi odd × 4	
xii odd × 5	
xiii odd × even	
xiv odd × odd	
xv odd × natural number (not 0)	
xvi odd × zero	



These dot patterns show how two six-patterns can be placed together to make rows and columns.

A 4 columns, 3 rows; 4 × 3 dots.
B 6 columns, 2 rows; 6 × 2 dots.
C 12 columns, 1 row; 12 × 1 dots.
D 3 columns, 4 rows; 3 × 4 dots.

The numbers 1, 2, 3, 4, 6, 12 are all *divisors* of 12. {1, 2, 3, 4, 6, 12} is the *set of divisors* of 12.

The numbers 6 and 12 are a *pair of factors* of 12; so are 3 and 4; and indeed 12 and 1 are too.

Exercise 2 **a** What do you notice about the arrangement of the pairs of factors in the set of divisors of 12 (provided the divisors are in order)?

b Arrange a set of dot-patterns to show all the factors of 24.

c Write down the set of divisors of each of

i 36
ii 60
iii 48
iv 72
v 144
vi 100

d Find all the pairs of factors of each of the numbers in (c).

Exercise 3 A *prime* number is one which divides exactly only by itself and 1.

a Write down the subset of prime numbers in the following set:
{15, 17, 27, 31, 29, 7, 5, 2, 9, 1}

b Are all prime numbers odd?

c Are all odd numbers prime?

d Write down the subset of the primes in the following set: $\{21, 121, 15, 39, 115, 57, 93\}$

e Do you think zero is a prime?

The word *product* is often used in mathematics. As in everyday life it means *something produced*. It could be the result of any operation whatever but in the arithmetic of numbers we keep the word to mean the *result of multiplication*. The product of 6 and 8 is 48. If I write $26 = 2 \times 13$ or $56 = 2 \times 2 \times 2 \times 7$ or $105 = 7 \times 5 \times 3$, I have expressed each of the numbers 26, 56, 105 as a *product of primes*.

Exercise 4 **a** Express the following as the products of primes:

i 24
ii 70
iii 154
iv 72
v 36
vi 60
vii 144
viii 115
ix 47
x 1000

b Are the following correctly expressed as the products of primes? If not, correct them.

i $27 = 3 \times 9$
ii $75 = 3 \times 15$
iii $96 = 3 \times 2 \times 2 \times 2 \times 2 \times 2$
iv $78 = 2 \times 39$
v $120 = 2 \times 2 \times 2 \times 3 \times 5$
vi $210 = 7 \times 5 \times 3 \times 2$

If we start from the set of natural numbers, we can invent some new sets. For instance, we can begin with

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots\}$

and add 5 to each element to make the set

$\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \ldots\}$.

Or we could multiply each element by 5 giving the set

$\{0, 5, 10, 15, 20, 25, 30, 35, \ldots\}$

Exercise 5 **a** Write down the first six members of the set we obtain when we add 3 to each of the natural numbers in turn.

b Write down the set we get by adding 10 to each of the natural numbers in turn.

c Write down the set obtained by multiplying the natural numbers in turn by 3. This set is called the set of *multiples of 3*.

d Write down the set of multiples of 7.

e What other name do you give to the set of multiples of 2?

f Add 1 to each element of the set of multiples of 2. What is your new set called?

g What is the set obtained by dividing the elements of the set of even numbers in turn by 2?

h When we count in 9's, we recite the elements of which set?

i The tables you learnt a long time ago are sets. Which sets are they?

j Write out the set of multiples of 16. Learn your '16 times table' – it will prove very useful to you.

k Write down the first eight members you get by dividing the elements of the following set by 3. $\{6, 12, 18, 24, \ldots\}$

l How can we obtain the following sets from the set of natural numbers? (In no case have we just left elements out!) Put down two more terms in each to show that you have really found out something about the sequences.

i $\{1, 3, 5, 7, 9, \ldots\}$
ii $\{3, 5, 7, 9, 11, \ldots\}$
iii $\{1, 7, 13, 19, 25, \ldots\}$
iv $\{4, 10, 16, 22, 28, \ldots\}$
v $\{5, 12, 19, 26, 33, \ldots\}$

m How many elements are there in the set we get by multiplying each of the elements of the set of natural numbers by zero?

n If M_2 is the set of multiples of 2, and M_3 the set of multiples of 3, and M_4 the set of multiples of 4 and so on, what are the sets

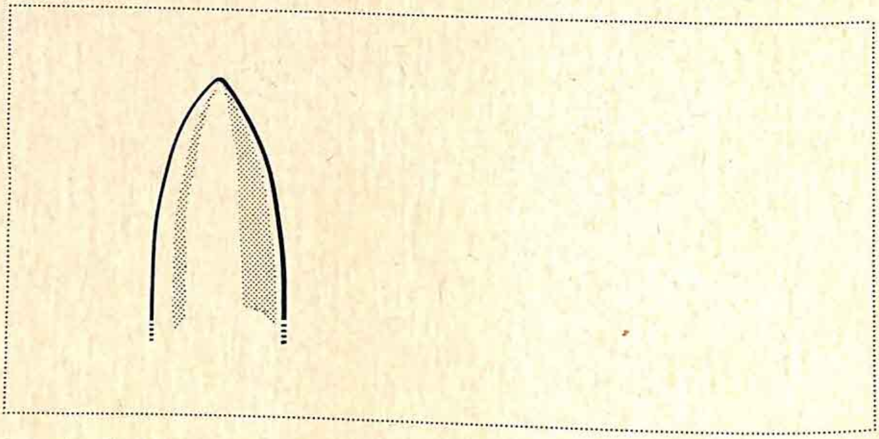
i $M_2 \cap M_3$
ii $M_2 \cap M_3 \cap M_4 \cap M_5$
iii $M_2 \cap M_3 \cap M_4 \cap M_5 \cap \ldots$?

o If we use n to stand for any 'any natural number' what is the intersection of all the sets M_n? (We can write this down as $\bigcap M_n$. Is it true that $\bigcap M_n = \{0\}$?)

p What is the difference between the empty set, $\emptyset$, and the set $\{0\}$?

Geometry

If you magnify the point of a pin you will see a shape something like this:

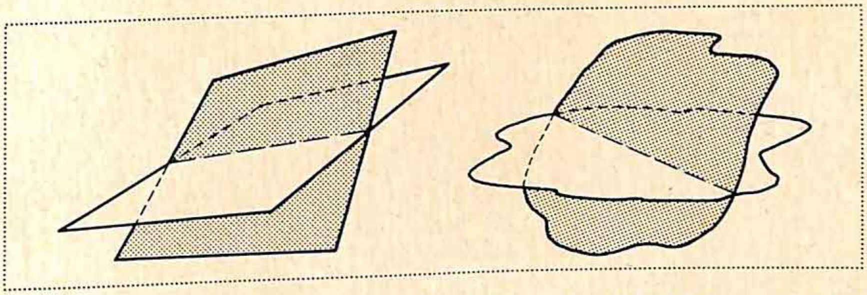


Imagine it sharpened under a microscope and then magnified again and then sharpened again. Imagine the process going on and on as long as you like. The tip is getting sharper and sharper and nearer to being what in geometry we call a *point*. (Will your pin ever really have a true point at its tip?) How big do you think a point is? You indicate a point with a '.' on paper. But is your dot really a point?

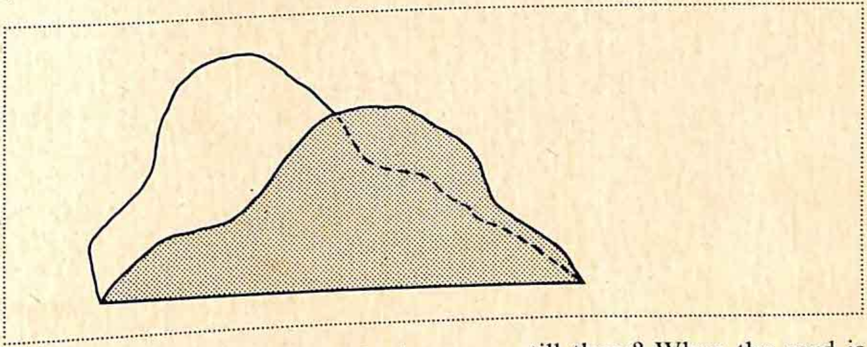
Sets of points are important: they are what geometry is about. For instance, the set of all points is what we call *space*. We shall look at certain special sets which are subsets of space. We call these subsets *geometrical figures*. Examples of these subsets are lines, plane surfaces, solids. An outline map of England or the drawing of a motor engine are figures in this sense. Even the shape of the actual motor engine itself could be called a geometrical figure. We only want to talk about some very simple shapes.

A *plane* is a completely flat surface and it goes on for ever in all directions. A table top is like part of a plane but is it completely flat? How would you discover whether it is or not? We shall come back later to this question.

Exercise 1 **a** A plane is a subset of the set of all points. What happens when two such subsets intersect? (It is best to think only of parts of the planes.) Along what sort of line do they intersect? Does the shape of the parts of the planes we think about make any difference to the line of intersection?



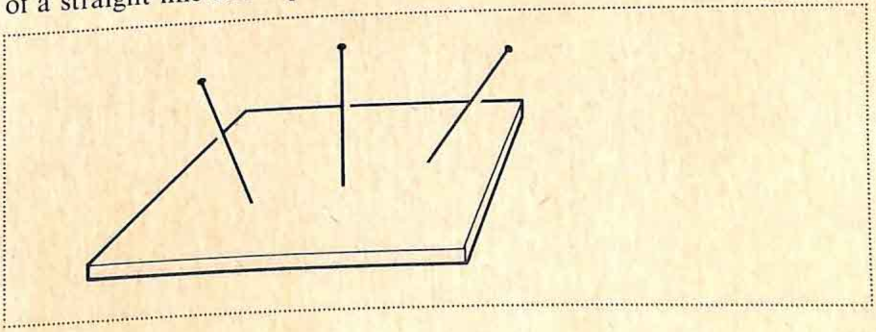
b Take an irregularly shaped piece of manila card and fold it once along a sharp crease. What does this tell you about the intersection of sets of points forming planes?



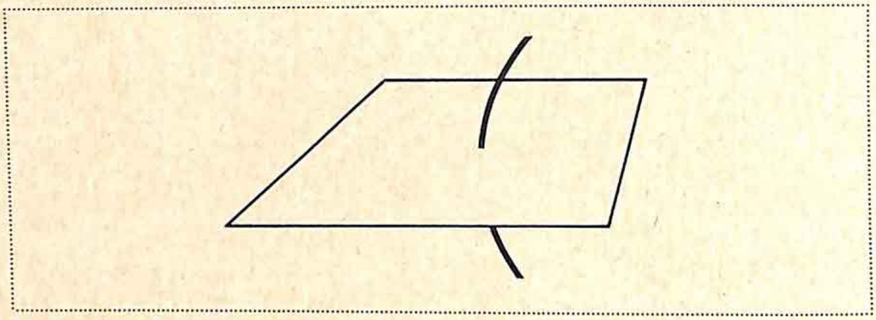
c Flatten out your card again. Is the crease still there? When the card is flat and forms one plane only, does the line of the crease lie in the plane? Did it lie in either of the planes when the card was folded? Could you say that the line is a subset of the points forming the plane with the card flat?

d Are all lines the same shape? Will two planes ever intersect to give any line which is not straight? Can you invent a definition of a straight line?

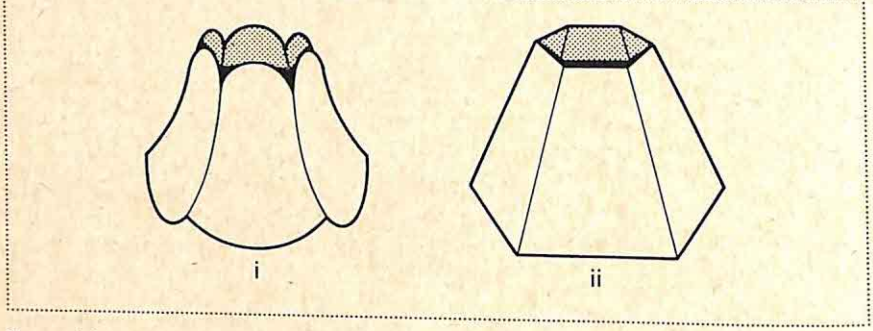
e A pin resembles a short piece of a straight line. Hold a pin in various positions on a flat table top. What does this tell you about the intersection of a straight line and a plane?



f What is the result of the intersection of a curved line and a plane?



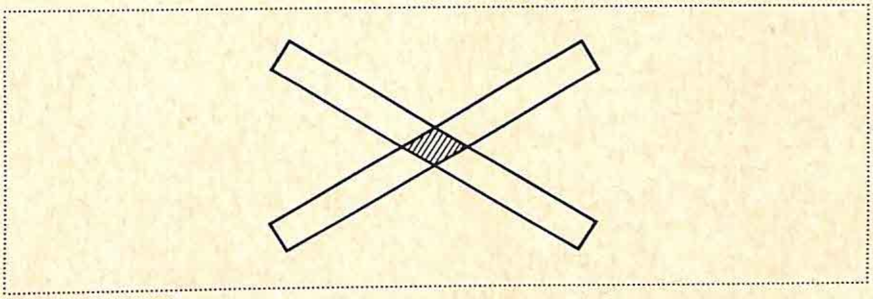
g Lamp shade i is made of six curved pieces while shade ii is made of six flat pieces. What shapes are the lines where the sections meet?

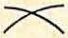
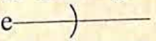
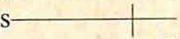


h Say which of the following statements are true (T) and which false (F), and explain your answers.

i a line is a subset of the plane
ii a line is a subset of space
iii the set of straight lines is a subset of the set of lines
iv the subsets of planes are called lines
v the subsets of lines are called points
vi a line is a set of points
vii two lines always intersect in a point
viii two lines can intersect in one point
ix two straight lines cannot intersect in two points only
x two lines can intersect in two points only
xi two straight lines which intersect in more than one point intersect in all their points.

i Place two rulers crossing one another as in the sketch. The shaded part is where they cross. This piece is part of both rulers: it is a common part. Imagine what happens to this common part if the rulers are made narrower and narrower. How big does this part become eventually? How wide are the rulers when they can become no narrower?



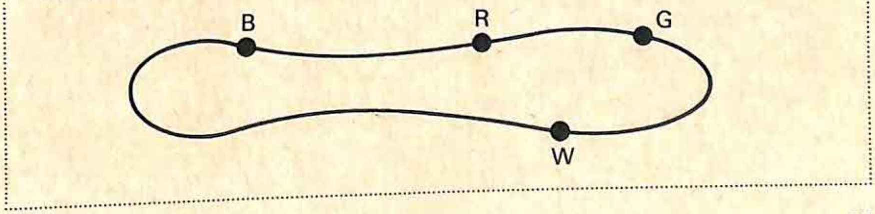
This gives us another way of indicating a point: by crossing two lines which may be or may not be straight. For instance we may have two curves meeting at a point:  or we may have a straight line and a curve  or even two straight lines  or .
Each line is a set of points 'side by side' and when two lines cross there is only one point in common.

j Pick up your ruler and look along the edge. What do you see? Is your ruler straight?

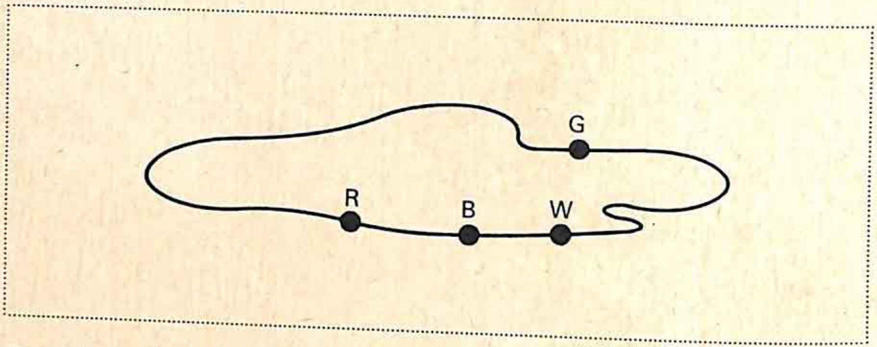
k If you take a steel ruler which has been very carefully machined, and examine it under a microscope, will it still be straight? Is it really straight at all?

The things we make are never perfect. The drawings we do are not either. Whenever we have an illustration of a geometrical idea, it is not exact. Quite often we make it deliberately inexact so that we do not make the mistake of guessing facts from the drawing. A great many times in this book and the next, you will do practical things to learn your geometry; you must always remember that the only way to be *certain* in mathematics is to reason about the facts. We can only imagine geometrical shapes or *figures*; any drawings we make are to help us to think about them more clearly.

Exercise 2 Imagine an elastic band lying on the page before you. Imagine also four dots of paint put onto it. They are four different colours: blue, red, green, white.



a Move the elastic band about, sideways, up and down, and changing its shape. Turn it too.



Does the order of the three points ever change?

b Turn the band over. Has the order changed?

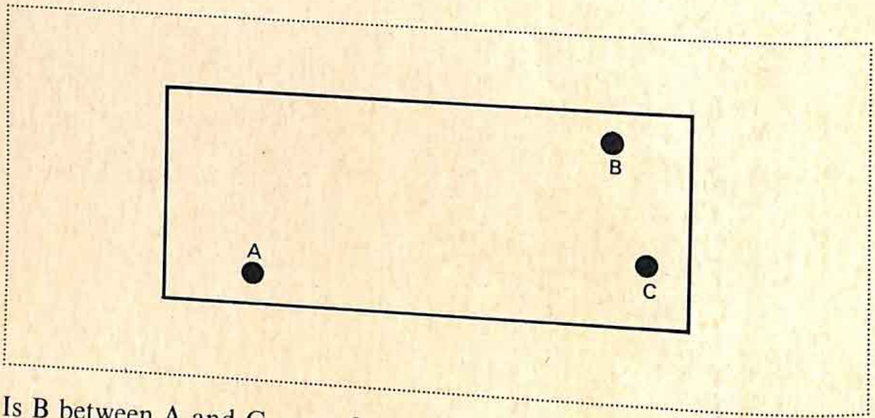
c Stretch the band. Does this change the order?

d If we agree that to decide the order we always begin at B, there are only two orders we can get by any of these changes. What are they?

e Can we ever find a way (without having the elastic band lifted off the paper at any point) in which we can get R between W and B?

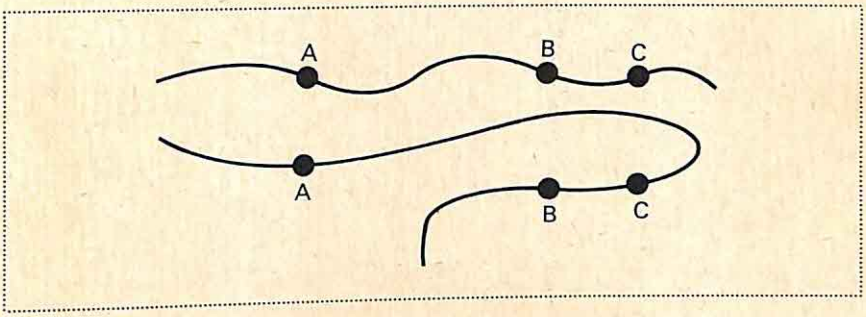
Exercise 3 **a** Where in the alphabet is the letter E? And where is it in relation to the letters D and F?

b



Is B between A and C or not?

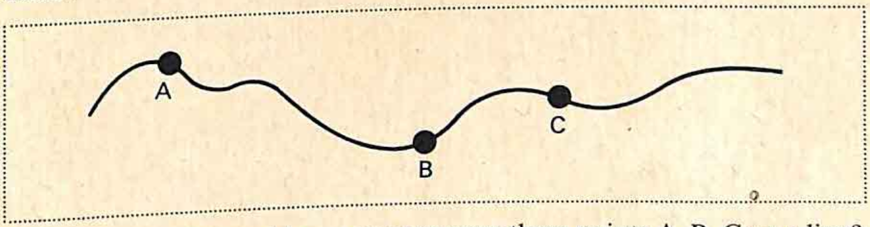
c Look at the line segments in these diagrams. Are the points A, B, C in the same order in each? What do you mean by order in this case? Is B between A and C in both?



d Copy the diagram of (b) four times. Can you use one of your diagrams to join up the points in the order A, B, C? And another in the order A, C, B? Can you use the third diagram to join the points in any other order? Can you use the fourth diagram to join the points so that B lies between A and C and also C lies between A and B?

e Would you say that order has anything to do with the idea of a line as a set of points?

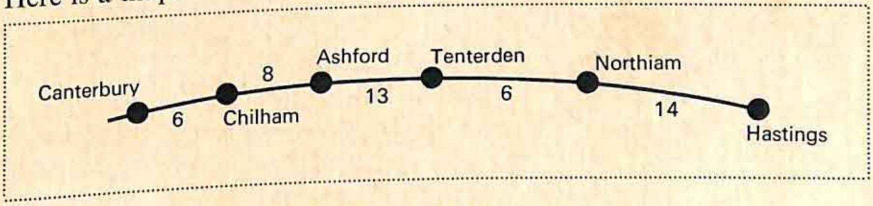
f How big are points? A line is a set of points in a given order. Imagine A and C close together in a line. Can you always find a point B between them?



g In how many ways could you arrange your three points A, B, C on a line?

h In how many ways could you arrange lines through four points?

Exercise 4 Here is a map.



a Does this map represent the shape of the road?

b Does it tell me the order in which I can expect the towns as I go from Canterbury to Hastings?

c Are the distances on the map to scale?

d How far is it from Canterbury to Hastings according to the map?

e Is the direction of the map correct?

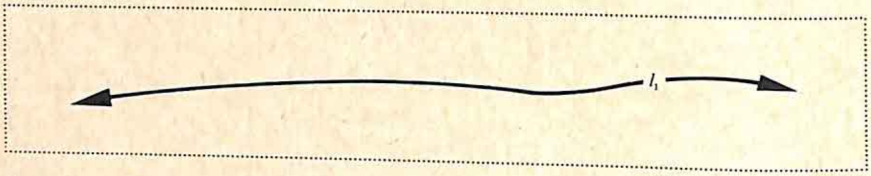
f Do you think that for driving from Canterbury to Hastings the map I have drawn would be better than an accurate one?

Exercise 5 **a** Draw a route from York to Barnsley via Pontefract in the same way that we did the one for the previous exercise.

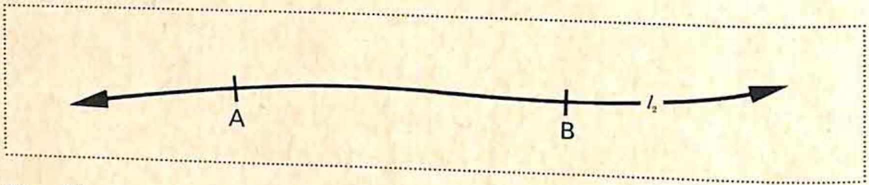
b Draw a similar map for a route in your own district.

c Examine a map of the London Underground. Are distances correct? Are directions correct? Does it tell you what you need to know in order to get from one place to another by Underground? What are the most important things it does tell you?

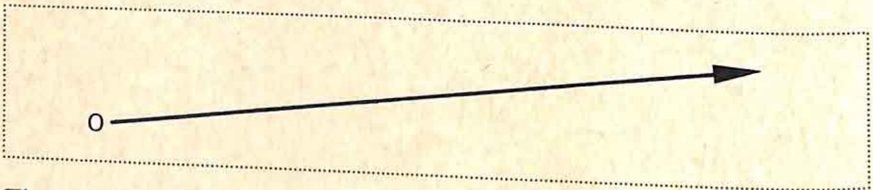
A *line* is a set which is *infinite* in length. That is it 'goes on for ever' in both directions. To show this, we can put arrowheads on it.



A piece of a line of *finite* length is called a *segment* of a line. AB is a segment of l_2.



We often want to talk about the part of a straight line on one side of a point.



The point we will call a *vertex* and name it O. The part of the line (which is infinite in one direction) we will call a *half-line*. When we want to think specially of the *direction* of a half-line we often call it a *ray*.

Exercise 6 In this exercise, all the lines and points are subsets of the same plane.

a Suppose a moving point A starts at a vertex O and travels out always going in the same direction. What sort of line would you call the path it traces out?

b Suppose two moving points A and B begin at O and move in exactly opposite directions, each keeping its own direction the same all the time.

i What shape is the path traced out?
ii What would you call it?
iii How long would be the traces in (b)?
iv Would they be the same length?

c Suppose a moving point A starts from a fixed vertex O_1 and another moving point B starts from a fixed vertex O_2 and both move in the same direction. Sketch the possible results. (There are two distinctly different cases.)

d A ray tells you two things: what are they?

e Two rays start from the same vertex and go in the same direction. What can you say about them?

f Two rays start at the same vertex and go in different directions. Draw a sketch of this situation.

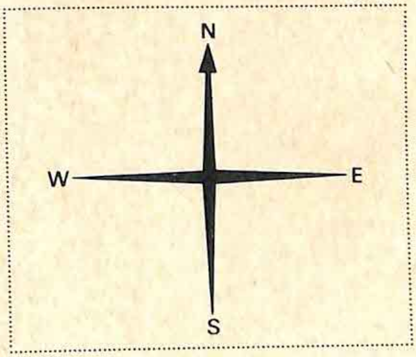
g Two rays start at different vertices and go in the same direction. Draw a sketch to show this.

h Two rays start at different vertices and go in different directions.

i Draw a sketch to illustrate this.
ii Do the rays cross?
iii Should you have drawn *two* sketches? If so, draw the other now.

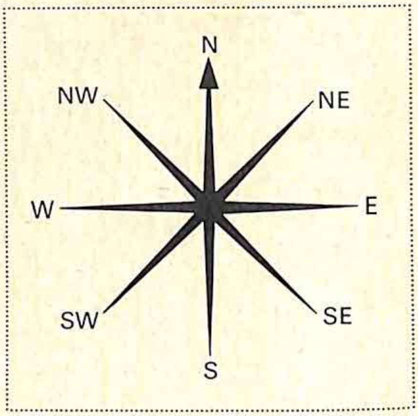
In (f) of the last exercise you drew two rays which were, as we say, 'at an angle'. We sometimes say that there is an angle *between* them. In (g) you drew two *parallel* rays.

Exercise 7 **a** What is meant by 'the points of the compass'?



b Imagine a ray from a vertex O pointing due north and another pointing due east. Each is said to be *perpendicular* to the other. Draw three sketches showing different pairs of perpendicular rays, using the directions E, S, W, N.

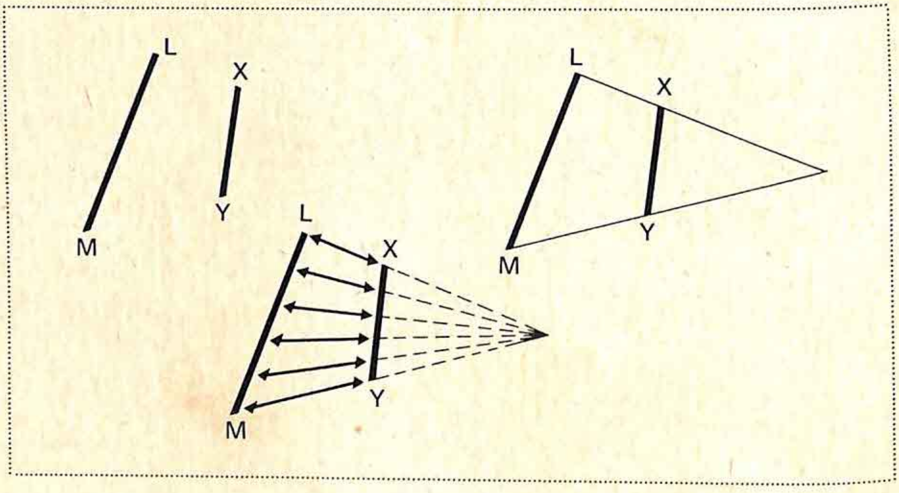
c Here is a compass illustrated with more of its points. Find four more pairs of perpendicular rays.



d Draw a diagram of the compass with eight more points on it. Find all the remaining rays which are perpendicular to each other.

Two perpendicular rays are at an angle. The angle in this case has a special name. It is called a *right angle*. Both your sets squares have right angles because we find them so useful in drawing.

Exercise 8 I have two line segments LM and XY. I am interested in mapping the points on LM into images in XY. The diagrams show how I am trying to do it.



a Can I find an image in XY for every point in LM by this method?

b Is there only one image for each point?

c Can I also find an image in LM for every point of XY?

d Is there only one image for each point?

e What sort of mapping is this?

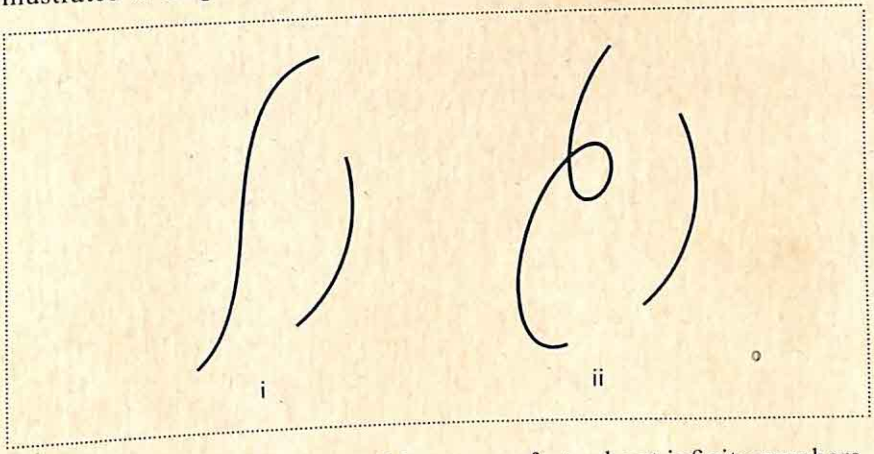
f When two sets are mapped in one-to-one correspondence, what can you say about the number of the sets?

g What can you say about the numbers of points in the two lines?

h Are the lines the same length?

i Could you use the same argument with any two straight line segments?

j Could you use the same argument with the pairs of curved line segments illustrated in diagrams i and ii?



You have discovered here one of the strange facts about infinite numbers. There are many more and you must never be surprised at odd results. After all infinity is an odd thing!

Section 7 **Replacement sets and graphs**

Exercise 1

- **a** What is a subset?
- **b** What is an empty set?
- **c** What are the possible subsets of the set $\{p, q, r\}$?
- **d** You should have included the set itself in the answer to (c). We call the other subsets *proper subsets*. Find all the different proper subsets of $\{a, e, i, o, u\}$.
- **e** Write out the set of whole numbers less than 10.
- **f** Partition the set you have just written into those which divide by 2 and those which do not.
- **g** Write down the set of odd numbers greater than 20 and less than 30.
- **h** Write out the set of multiples of 7 which are greater than 50 and less than 80.
- **i** Write out the set of multiples of 17 which lie between 117 and 127.
- **j** What is the set of multiples of 19 which lie between 120 and 130?

Exercise 2

Write out a description of the following sets.

- **a** $\{9, 12, 15, 18\}$
- **b** $\{13, 26, 39, 52\}$
- **c** $\{13, 26, 39, 52, \ldots\}$
- **d** $\{20, 30, 40, 50, 60\}$

Exercise 3

- **a** If I say I have less than 6d, how much money do you think I might have?
- **b** What is the greatest length possible for a piece of string known to be a whole number of inches but less than 2 feet long?
- **c** Does *less than* ever include the end point?
- **d** Write down the whole numbers between 3 and 6. Did you think of including either 3 or 6? Explain.
- **e** How many elements are there in the following sets of whole numbers?
  - i the set of even numbers between 7 and 23
  - ii the set prime numbers between 22 and 30

iii the set of odd numbers between 28 and 30
iv the set multiples of 27 between 60 and 65
v the set of odd numbers greater than 65
vi the set of odd numbers
vii the set of natural numbers

We are now going to use a box like this: $\square$; or this: $\triangle$; or this: $\triangledown$. Into the box we can put any number from the set of natural numbers. The set of natural numbers we will say is the *universe* for this series of problems. Here is a statement with a box in it: $5 < \square < 10$. If you are asked whether the statement is true or false you will not be able to say because the statement will not be complete until the box is filled. This is like asking if the statement 'He is a film director' is true or not without being told who 'he' is! As soon as you are told it becomes *possible* for you to answer the question (although of course you may not actually know the answer). Before, it was quite *im*possible. When we fill our box with a number from the universe we shall be able to say whether the statement is true or not. For instance: $5 < \boxed{13} < 10$ is false; or: $5 < \boxed{7} < 10$ is true. There will be a set of numbers which will make your statements true in each case.

Exercise 4 Find the sets of values which make these statements true.

a $7 < \square < 11$
b $4 < \square < 13$
c $9 < \triangledown \leqslant 14$
d $16 \leqslant \triangle < 19$
e $3 > \square > 1$
f $8 \geqslant \triangledown > 2$
g $5 > \triangledown \geqslant 1$
h $15 \geqslant \triangle \geqslant 11$
i $7 > \square > 6$
j $0 < \triangle < 5$
k $\triangle + 4 = 6$
l $\triangle + 6 = 13$
m $5 + \triangledown = 21$
n $31 - \square = 15$

Exercise 5 Write down the sets of natural numbers which make these statements *false*.

a $2 < \square \leqslant 5$
b $6 > \square > 4$
c $\triangledown > 0$
d $\triangle < 0$
e $\triangle + 10 = 20$
f $\triangledown - 15 = 0$
g $\square - 0 = 0$

The sets which you have been writing are called either *replacement sets* or *solution sets.* We often refer to *solution sets* when the statement contains an equals sign only.

Instead of boxes, we can use letters. We shall use small letters like x, y, p, q, e, z, k and so on. It does not matter which letters we use, the effect is the same. They are easier to write than the boxes and that is the only reason for using them at this stage. These small letters are called *variables.*

Exercise 6 **a** Find the replacement set for x if x is any even number between 21 and 30.

b n is any natural number below 50 and it is both a multiple of 3 and of 5.

c $y < 8$ and is a whole number. Find its replacement set.

d $c > 20$ and is a prime number. Find the four least members of the replacement set.

e Would you say that the whole set in d is finite or infinite?

f '$x > 3$ and $x < 8$' is shortened to '$3 < x < 8$'. Find its replacement set.

g $5 \leqslant p < 14$ and p is odd. Find its replacement set.

h | is read 'such that'. What does the following mean?
$\{q \mid 0 < q < 6; q \text{ is a natural number}\}$.
Translate it into words.

i Find the replacements set in (h).

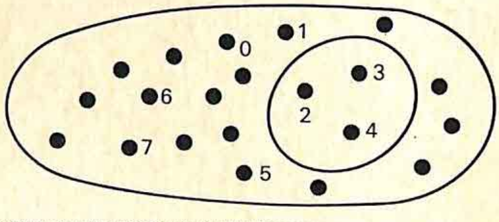
j What is the replacement set for the variable in the set
$\{a \mid 90 < a \leqslant 102; a \text{ is a multiple of } 3\}$?

k Find the replacement set of $\{x \mid x > 4; x \text{ is a multiple of } 3\}$.

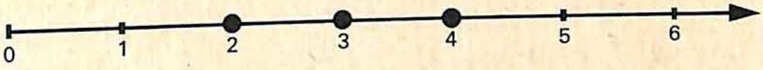
l Find the replacement set of $\{z \mid 5 < z < 8; z \text{ is a multiple of } 5\}$.

m What is the replacement set of $\{q \mid q = 12\}$?

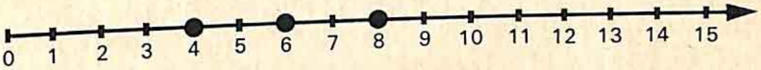
As we have seen before, we can draw diagrams of sets. For instance the set $\{2, 3, 4\}$ might be drawn like this:



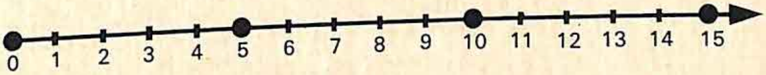
As you also will have noticed, some sets contain numbers, and numbers can be arranged in order. When a set consists of elements in order, we can make another representation or *model* of the set. All we need is something we can see which also can be arranged in order: for instance, points in a line. Here is a model of the same set and the universe of natural numbers from which it was taken.



A model of this kind is called a *graph*. Here are the graphs of two other sets, both taken from the set of natural numbers.



{4, 6, 8, 3}



{0, 5, 10, 15}

Exercise 7 Draw graphs of the following sets:

a {2, 4, 6, 8}

b {3, 6, 9, 12, 15}

c {0, 1, 3, 6, 10}

d $\{1, 2, 3, 5, 7\} \cap \{4, 5, 6, 7, 8\}$

e $\{1, 2, 3, 5, 7\} \cup \{4, 6, 8\}$

f the set of even numbers between 6 and 12

g the set of multiples of 5 greater than 0 and less than 20

h the set of multiples of 3 which are also multiples of 4

i the set of odd numbers

j the set of natural numbers

k the set of x such that x lies between 5 and 8 and x is a natural number

l $\{x \mid 3 < x < 9;\ x \text{ is a natural number}\}$

m $\{y \mid 6 < y \leqslant 7;\ y \text{ is a natural number}\}$

n $\{z \mid 0 \leqslant z < 6;\ z \text{ is a natural number}\}$

o $\{p \mid 5 \geqslant p \geqslant 2;\ p \text{ is a natural number}\}$

p $\{q \mid 8 > q > 4;\ q \text{ is a natural number}\}$

Exercise 8 In this exercise we shall write down expressions like $2a$, $3x$, $4y$ and so on. Each will be a number followed by a letter. To find out what they mean read each of the statements below very carefully. Then do the exercises below.

$\{2a \mid 1 < a \leqslant 5;\ a \text{ is a natural number}\} = \{4, 6, 8, 10\}$
$\{3x \mid 0 < x < 4;\ x \text{ is a natural number}\} = \{3, 6, 9\}$
$\{5y \mid 4 \leqslant y \leqslant 7;\ y \text{ is a natural number}\} = \{20, 25, 30, 35\}$

a What do you think $2a$ means? Is it $2+a$, $2-a$, $2\times a$, $2\div a$?

b How else could you write $3x$?

c Does $5y$ mean $y+y+y+y+y$?

d We often miss out multiplication signs in mathematics when it does not cause confusion to do so. Is this what we have done here?

e Write each of the following in at least one other way. If you can find more ways then put them all down.

i $2\times a$
ii $3b$
iii $17\times p$
iv $4x$
v $41\times z$
vi $a\times b$
vii cd
viii $4\times r$
ix $2\times 3a$
x $3\times 2a$
xi $5a\times 6$
xii $4\times 3\times a$
xiii $5\times 2\times b$
xiv $7c\times 2$
xv $9\times d\times e$
xvi $7\times fg$
xvii $6h\times j$
xviii $p\times q\times r$

Exercise 9 Write out the replacement sets of the following and draw their graphs. You will find two colours useful.

a $\{3x \mid 2 < x < 5;\ x \text{ is a natural number}\}$

d $\{2y \mid 0 \leqslant y < 6;\ y \text{ is a natural number}\}$

c $\{5z \mid 1 < z \leqslant 3;\ z \text{ is a natural number}\}$

d $\{2a \mid 4 < a < 6;\ a \text{ is a natural number}\}$

We have been leaving out multiplication signs in expressions like $4 \times y$ so that we write $4y$. We can also leave out the sign in an expression like $\frac{1}{2} \times y$ and write $\frac{1}{2}y$.

Exercise 10 Write out the replacement sets of the following and draw graphs of them.

a $\{x \mid 4 < x \leqslant 6;\ x \text{ is a natural number}\}$

b $\{x \mid 4 < 2x < 8;\ x \text{ is a natural number}\}$

c $\{x \mid 0 < x < 5;\ x \text{ is a natural number}\}$

d $\{x \mid 1 \leqslant 3x < 10;\ x \text{ is a natural number}\}$

From now on we shall say '$x \in \mathrm{N}$' instead of 'x is a natural number'. How do we read out '$x \in \mathrm{N}$'?

Exercise 11 Write out the following sets and draw graphs of them.

a $\{x \mid 1 \leqslant x < 8;\ x \in \mathrm{N}\}$

b $\{x \mid 7 \geqslant x > 3;\ x \in \mathrm{N}\}$

c $\{x \mid x = 2z;\ 6 < z < 10;\ z \in \mathrm{N}\}$

d $\{x \mid x = 3y;\ 0 \leqslant y < 5;\ y \in \mathrm{N}\}$

e $\{p \mid p = 5r;\ 1 < r \leqslant 7;\ r \in \mathrm{N}\}$

f $\{q \mid q = 3r;\ 5 \leqslant 3r \leqslant 7;\ r \in \mathrm{N}\}$

g $\{s \mid s = 3r;\ 5 < 2r < 7;\ r \in \mathrm{N}\}$

h $\{t \mid t = 2r;\ 0 < r;\ r \in \mathrm{N}\}$

i $\{a \mid a = 3b;\ 0 \leqslant b;\ b \in \mathrm{N}\}$

j $\{x \mid x = \frac{1}{2}y;\ 0 \leqslant y < 6;\ x \in \mathrm{N}\}$

k $\{x \mid x = \frac{1}{2}y;\ 0 < y;\ y \text{ is a multiple of } 2;\ y \in \mathrm{N}\}$

l $\{x \mid x = \frac{1}{4}y;\ 0 \leqslant y;\ x \in \mathrm{N}\}$

m $\{p \mid p = \frac{1}{6}q;\ q \text{ is a multiple of } 6;\ p \in \mathrm{N}\}$

n $\{r \mid r = \frac{1}{2}s;\ s = \frac{1}{2}t;\ 11 < t < 13;\ t \in \mathrm{N}\}$

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Exercise 12 Write out the sets which satisfy the following.

a $\{y \mid y = x+4;\ 0 < x < 5;\ x \in \mathrm{N}\}$

b $\{x \mid x = y-6;\ 6 \leqslant y < 8;\ y \in \mathrm{N}\}$

c $\{y \mid y = x+5;\ 1 < x < 8;\ x \in \mathrm{N}\}$

d $\{x \mid x = y-8;\ 0 < y < 4;\ y \in \mathrm{N}\}$

e $\{z \mid z+4 = 6;\ z \in \mathrm{N}\}$

f $\{y \mid y-3 = 8;\ y \in \mathrm{N}\}$

g $\{x \mid x+5 = 12;\ x \in \mathrm{N}\}$

h $\{y \mid y+4 = 1;\ y \in \mathrm{N}\}$

i $\{x \mid x-6 = 2;\ x \in \mathrm{N}\}$

j $\{z \mid z+4 = 10\}$

k $\{k \mid k-6 = 8\}$

l $\{m \mid m-8 = 9\}$

m $\{n \mid 2n = 12\}$

n $\{p \mid 3p = 6\}$

o $\{q \mid 5q = 10\}$

p $\{r \mid 7r = 14\}$

q $\{s \mid 15s = 135\}$

r $\{a \mid 5a = 24;\ a \in \mathrm{N}\}$

s $\{b \mid 3b = 12\}$

t $\{p \mid \frac{1}{4}p = 2\}$

u $\{a \mid \frac{3}{8}a = 1\}$

v $\{x \mid \frac{3}{4}x = 3\}$

Exercise 13 **a** In which two ways can you write down a set?

b How can you make a sketch to represent any set?

c What do you call the sort of sketch you have used to illustrate a set where the elements are shown in a special order?

d What are the signs for 'less than' and 'greater than'?

e On the number line (drawn across the page) does 'less than' mean 'to the right of' or 'to the left of'?

f If a statement has a box in it without a number in, it is sometimes said to be *open*. Can you say whether an open statement is true or not?

g What name do we give to the letter x in $\{x \mid x \ldots$ etc.$\}$?

h Why do we give it this name?

i In what other ways can we write down $3y$?

j What symbol have we been using for 'the set of natural numbers'?

k How do you abbreviate 'x is a member of the set of natural numbers'?

l Why did we suddenly stop writing '$x \in N$' in Exercise 12?

m Draw a number line and mark on it about six natural numbers. Now write a question like one of those in Exercise 12 and give it to your neighbour to do. Your mark will be for getting the *question* right!

Section 8 **More about numbers**

Exercise 1 **a** What do you call the numbers you use for counting people, motor-cars, piles of stones?

b Look at the set below and then write down a subset consisting entirely of natural numbers.

$\{1, 5\frac{1}{4}, 3{\cdot}1, \sqrt{2}, 3, 0, 4^2, 12, \triangle, \frac{25}{2}, e, 7, \theta\}$

c Did you include 0 in your subset? Can there be 0 people in a room or 0 cars in the street or 0 piles of bricks in a builder's yard?

d Is 0 a counting number?

Exercise 2 **a** What are the numbers of the two sets A and B?

$A = \{\triangle, \bigcirc, \diamondsuit\}$

$B = \{\mathfrak{R}, \mathfrak{S}, \mathfrak{T}, \mathfrak{U}\}$

b Do the two sets A and B overlap?

c What sort of set is $A \cap B$?

d List the set $A \cup B$.

e What is the number of this new set?

f Complete the following:

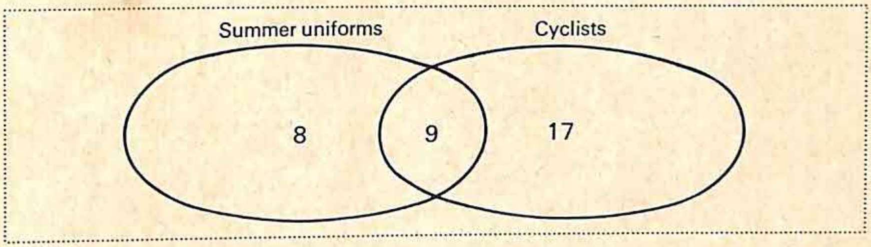
i $n(A) =$ ii $n(B) =$ iii $n(A \cup B) =$

g Is the statement $n(A)+n(B) = n(A \cup B)$ true in this case?

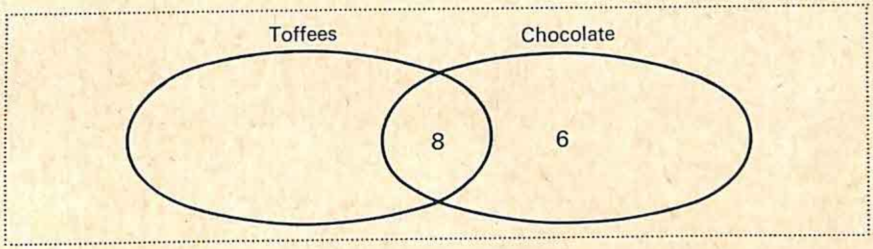
h Is the statement in g always true for *any* sets? Explain. Would it have been true if your answer to b above had been different?

i Draw a Venn diagram to illustrate each of the two kinds of situations.

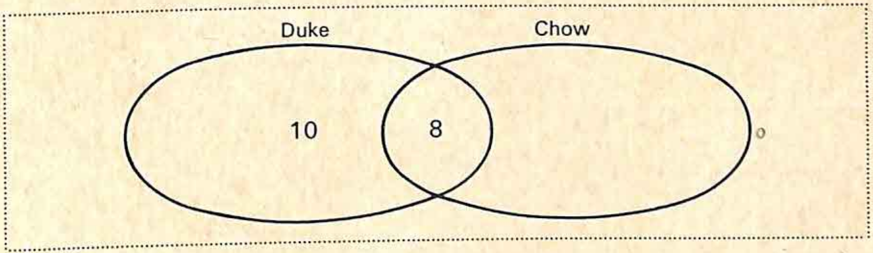
Exercise 3 **a** In Hotrod School there is a class of 34 children. Twenty-six come to school by bicycle. Seventeen are wearing their summer uniforms. What is the least number of cyclists wearing summer uniforms?



b Eighteen members of the class have brought sweets with them today. Fourteen of them have chocolate and 8 of these have toffees as well. All the rest have only toffees. How many have toffees alone?



c The two TV comedians most popular in the class are Dick Duke and Charley Chow. Ten like Duke but not Chow. Six like Chow but not Duke. Eight like both. How many members of the class like one or the other?



d Here is a set of questions made up by children in Shropshire, England. For each of them draw a Venn diagram showing two intersecting sets, and answer the question asked.

- i Of 38 girls in 3C, 22 like needlecraft and 22 like domestic science, How many like both?
- ii There are 37 boys in 4B. Twenty-three of them like woodwork and 21 like metalwork. None of them dislikes both. How many like both?
- iii One-hundred-and-twenty-one women who smoke were interviewed. One-hundred-and-nineteen of them said they smoked filter-tipped cigarettes and 37 smoked untipped cigarettes. None of them said they smoked cigars or pipes. How many never smoked untipped cigarettes?

iv Of 178 men cigarette smokers, 76 smoked filter tipped and 122 smoked plain cigarettes. How many smoked both?

v Six-hundred-and-eighteen women were asked about their washing for a market survey. Three-hundred-and-ninety-seven used detergents at least part of the time while 407 used soap powders and flakes. How many of them sometimes used one and sometimes the other?

vi On a housing estate, there are 68 front gardens with lawns, and 78 with flower beds. Sixteen have only flower beds and there are no neglected gardens. How many houses are there with front gardens?

vii In a group of 180 children, 179 are right-handed, and 48 are left-handed. How many are ambidextrous?

viii There are 39 mathematics exercise books in class 1A. Thirty-two of them have some marking done in pencil while 37 have some done with coloured ball pen. How many are marked in pencil only and how many in ball pen only?

ix Three-hundred-and-seventy-nine people on an estate watch television regularly and 519 listen to the wireless. There are 540 people with either TV or radio; how many of them do not see TV regularly?

x Of 260 customers in a certain sweet shop, 170 said they liked boiled sweets and 127 said they liked soft sweets. How many like both?

xi Seventy-three girls in a school like hockey, and 148 like swimming. There are 250 girls in the whole school of whom 50 like neither swimming nor hockey. How many like both?

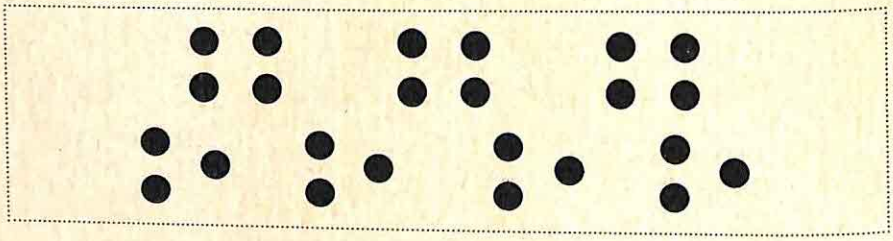
xii Out of the 555 people at Buckland (all of whom like either the theatre or the cinema) 253 say they like the theatre and 371 say they like the cinema. How many like both?

xiii In a school of 530 children, 271 say they like drinking milk and 491 say that they like drinking lemonade. Twenty do not like either. How many like milk but not lemonade?

xiv I know 25 children, 12 of whom have brothers and 17 have sisters. None is an only child. How many have both brothers and sisters?

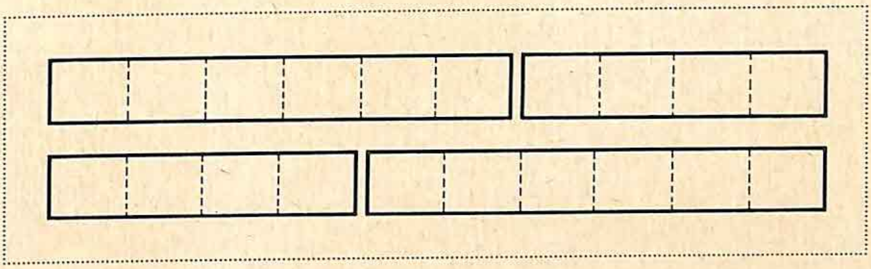
xv A Youth Club of 144 members is organizing two trips for next Friday: one to a dance and one to an ice-show. 90 of the members like dancing and 70 like watching ice-skating. Will it be possible to make two equal sized parties on Friday so that everyone goes to one or the other and no-one goes to something he or she dislikes?

e



i Are 3 × 4 and 4 × 3 really the same?
ii Do they mean the same?
iii Are they equal?
iv Does the same hold for 7 × 8 and 8 × 7?
v Is the same true for *any* two natural numbers?
vi What sort of saving does this make in the learning of the multiplication tables?

f



i Are 6 + 4 and 4 + 6 really the same?
ii Are they equal?
iii Does the same thing hold for *any* pair of natural numbers?

× is an *operation.* It tells us to do something or rather that something is done. 3 × 4 means 'Start with 4; put it down 3 times and add'. × is an operation which works both ways to give the same result. So 4 × 3, which means 'Begin with 3, put it down 4 times and add', comes to the same value in the end.

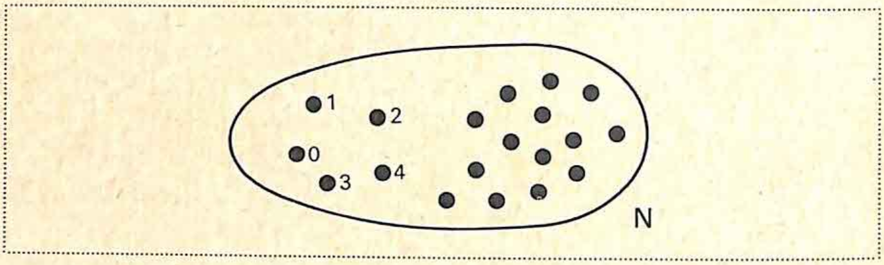
Exercise 4 **a** i Is + an operation?
ii Does + work both ways?

b i If you have breakfast and then get up, is the *result* the same as if you had got up and then had breakfast?
ii If you put on your socks and then your shoes is the result the same as if you put on your shoes and then your socks?
iii Do all operations work both ways?

c An operation which works both ways is said to be *commutative*. Make up four examples of commutative operations from everyday life. (The operation in (b) i and ii is that of *combining* two things.)

d Make up four examples of non-commutative operations from everyday life.

When we want to say that any two natural numbers can be multiplied either way round to give the same result, we can do it much more simply. We want to talk about the set of natural numbers, so we will call that N.



We want to talk about two numbers, any two numbers selected from that set, so we will use two *variables*, a and b. The variables can *range* over the whole of N. The statement we want to make is this:

For all numbers a and b chosen from N, $a \times b = b \times a$.

$$\forall\ a\ ,\ b\ \in\ \mathrm{N} : a \times b = b \times a.$$

The second line is the first written in symbols. Read it out as: For all a and b which are elements of N, etc.

Exercise 5 **a** Write down a statement in words which tells you a similar fact about addition.

b Translate your statement into symbols.

c Read out the symbols.

Exercise 6 **a** Say whether or not these statements are true. (The brackets are instructions to work out that bit first.)

i $7 \times (5 \times 3) = (7 \times 5) \times 3$
ii $8 \times (4 \times 9) = (8 \times 4) \times 9$
iii $(2 \times 19) \times 3 = 2 \times (19 \times 3)$
iv $6 \times (5 \times 4) = (6 \times 5) \times 4$
v $a \times (b \times c) = (a \times b) \times c$ where a, b and c are any natural numbers.
vi $\forall\, a, b, c \in \mathrm{N} : a \times (b \times c) = (a \times b) \times c$

b The law you have just stated is the *associative law for multiplication.*
i Is there an associative law for addition?
ii If there is, write it out in words.
iii Now translate ii into symbols.

c Check these two statements by arithmetic.
i $5 \times (4 \times 7) = 4 \times (5 \times 7)$
ii $5 + (4 + 7) = 4 + (5 + 7)$

The last question suggests that you can multiply or add natural numbers in any order.

We know now that this is true of a particular set of three natural numbers. We could go on trying different sets of natural numbers. How long would it take to do them all? We shall be doing them all – all at once! But first we will try one but we shall not work anything out.

$5 \times (4 \times 7) = (5 \times 4) \times 7$ by the associative law
$(5 \times 4) \times 7 = (4 \times 5) \times 7$ using the commutative law
$(4 \times 5) \times 7 = 4 \times (5 \times 7)$ using the associative law again

We did no arithmetic but we used two laws which we know to be true. Now let us do it for any three natural numbers. Remember a, b, c can be any natural numbers (and they might all be the same one!).

$a \times (b \times c) = (a \times b) \times c$ by the associative law
$(a \times b) \times c = (b \times a) \times c$ by the commutative law
$(b \times a) \times c = b \times (a \times c)$ by the associative law

This means that whatever the numbers are, you can change the order in this particular way. You can prove that all of the twelve possible arrangements of the three numbers are equal to one another.

Exercise 7

a Show that $a \times (b \times c) = a \times (c \times b)$

b Show that $a \times (c \times b) = (a \times c) \times b$

c Show that $(a \times c) \times b = (c \times a) \times b$

d Write out a full proof that $a \times (b \times c) = (c \times a) \times b$

e Would you say that everything we have said about multiplication since Exercise 6 also applies to addition? Explain.

f Prove that $a + (b + c) = c + (a + b)$

Exercise 8

a Can we always do operations in any order?

b Subtraction is an operation. Is it commutative? Give examples.

c Division is an operation. Is it commutative? Explain.

d Can we add and subtract in any order? That is, is $12 + 3 - 4 = 12 - 4 + 3$ and so on?

e Can you see a kind of problem where it would be very difficult to do the subtraction first as in the right hand side of (d)?

f Find the answers to the following by the shortest means you can.

i $7 - 9 + 3$
ii $15 - 18 + 3$
iii $27 - 36 + 18$
iv $41 + 17 - 11$
v $76 + 18 - 26$
vi $115 - 76 + 81$
vii $1002 - 22 + 67$
viii $197 + 15 - 99$

Exercise 9 In this work remember that brackets, (...), tell you to do the work inside them first.

a Work out i $7 \times (2+8)$
and ii $(7 \times 2)+(7 \times 8)$

b Find i $5 \times (4+6)$
and ii $(5 \times 4)+(5 \times 6)$

c Simplify i $2 \times (9+7)$
and ii $(2 \times 9)+(2 \times 7)$

d Calculate i $10 \times (12+8)$
and ii $(10 \times 12)+(10 \times 8)$

e Compare i $8 \times (10+7)$
with ii $(8 \times 10)+(8 \times 7)$

f Use your normal method to work out
i 9×28 ii 8×57 iii 7×86

g There is a general rule which applies to all these questions. What is it?

h Explain why
$$\begin{array}{r} 57 \\ \times\ 4 \\ \hline 228 \end{array}$$
gives the same result as $(4 \times 50)+(4 \times 7)$

In the last exercise you have been using a general law like the others you have met in this section. You have not *proved* that these laws are true, but you can test them out with any numbers which you care to use and each time you will find that the laws work. (Satisfy yourself that this is true.) We shall from now on accept the laws as true.
Here they all are:

$\forall x, y \in \mathrm{N}: x+y = y+x$
$\forall x, y \in \mathrm{N}: x \times y = y \times x$
$\forall x, y, z \in \mathrm{N}: x+(y+z) = (x+y)+z$
$\forall x, y, z \in \mathrm{N}: x \times (y \times z) = (x \times y) \times z$
$\forall x, y, z \in \mathrm{N}: x \times (y+z) = (x \times y)+(x \times z)$

Exercise 10 **a** Translate all of these five laws into words

b Make up examples to illustrate each of them

Exercise 11 Check the following statements and put YES or NO in each column against each statement.

		This statement is true	A statement like this is true of any three natural numbers
a	$6 \times (5 \times 4) = (6 \times 5) \times 4$		
b	$(12+3)+8 = 12+(3+8)$		
c	$89 \times (12 \times 2) = (89 \times 12) \times 2$		
d	$113+(276+81) = (113+276)+81$		
e	$4 \times (5+3) = (4 \times 5)+(4 \times 3)$		
f	$8+(6 \times 2) = (8+6) \times (8+2)$		
g	$47 \times (18+6) = (47 \times 18)+(47 \times 6)$		

In Exercise 11 e, f, g we put brackets round the multiplications to show that we should do these first. We usually agree, however, that multiplication is to be done before addition if there are no brackets, and we only put brackets in if we want to instruct the reader to do some addition first.

Exercise 12 **a** Is the following statement true? $27 \times (18+13) = 27 \times 18 + 27 \times 13$?

b Use what you have found out earlier in this section to write the following in brackets. Work it out by the quickest method.

$59 \times 4 + 59 \times 6$

c Do the same with $271 \times 64 + 271 \times 36$.

d Now try $3942 \times 31 + 3942 \times 19$.

Exercise 13 **a** i Can we add up in any order? ii Can we multiply in any order?

b Are the results of the following pairs of calculations the same? Do not work them out.

i $75+13+56$ and $56+75+13$.
ii $75 \times 13 \times 56$ and $13 \times 56 \times 75$.
iii $75+13 \times 56$ and $75 \times 56+13$.
iv $(75+13) \times 56$ and $75 \times 56 + 56 \times 13$.

Exercise 14 Examine the following statements and say whether they are true or false. Only work them out if you cannot tell without doing so.

a $25 \times 14 + 14 \times 60 = 14 \times 85$

b $371 + 157 \times 526 = 526 \times 157 + 371$

c $371 \times 157 + 526 = 526 \times 371 + 157$

d $54 \times 50 \times 2 = (54 \times 50) \times 2$

e $(54 \times 50) \times 2 = 54 \times (50 \times 2)$

f $29 \times (36 + 51) = 36 \times 29 + 29 \times 51$

g $29 + (36 \times 51) = (29 + 36) \times (29 + 51)$

h $17 + 29 + 34 = 29 + 17 + 34.$

Exercise 15 **a** We sometimes leave out multiplication signs before brackets; we never leave out addition signs. What does $36(2+28)$ mean? Work it out.

b Work out $59(87+13)$

c We sometimes use a dot on the line to represent multiplication, so that $3 \,.\, 5$ means 3×5. In America the dot is placed higher up, eg. $3 \cdot 5$. This has another meaning in Great Britain. See Section 27: Numbers in their places. Work out the following by the shortest method you can find.

i $56 \,.\, 8 + 7 \,.\, 44$ iii $4(97 \times 25)$ v $89 \,.\, 5 + 9$

ii $100(17+6)$ iv $89 \,.\, 9 + 5$ vi $960 \times 50 \times 14$

Exercise 16 Explain the connexions between the following:

a 4×7; $7+7+7+7$; 7×4; $4+4+4+4+4+4+4$

b $27 + 16 = 43$; $43 - 27 = 16$; $43 - 16 = 27$

c $4 \times 7 = 28$; $28 \div 4 = 7$; $28 \div 7 = 4$; $27 - 7 - 7 - 7 - 7 = 0$;
$28 - 4 - 4 - 4 - 4 - 4 - 4 - 4 = 0$

Section 9 **Points, lines and planes again**

Exercise 1 Think of a plane. Remember how big it is. Call the plane P. In it lies a line *l*.

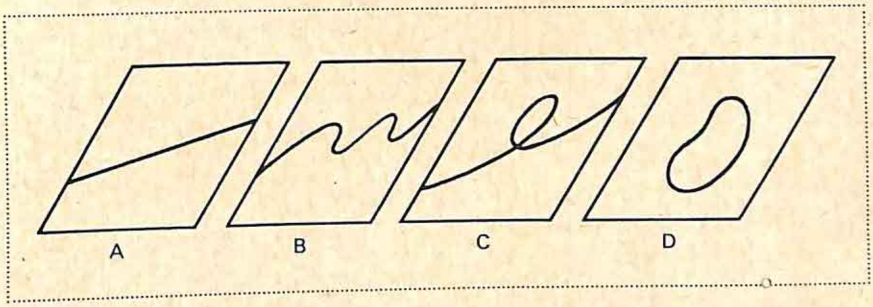
a What does the statement $l \subset P$ mean?

b There are two other lines l_1 and l_2 also lying in the plane. Write down statements to say that

i l_1 lies in the plane P

ii l_1 intersects l_2 in no points

Exercise 2 The diagrams A, B, C, D illustrate planes, and each contains a line. We call the four planes P_A, P_B, P_C, P_D and the corresponding lines l_A, l_B, l_C, l_D.



a P_A is partitioned into three subsets. Describe them.

b Into how many parts are the planes (i) P_B, (ii) P_C, (iii) P_D, partitioned?

c What do you think might be meant by a half-plane? (Remember what we meant by a half-line.)

d Copy the diagram which shows what you think is a 'simple closed curve'.

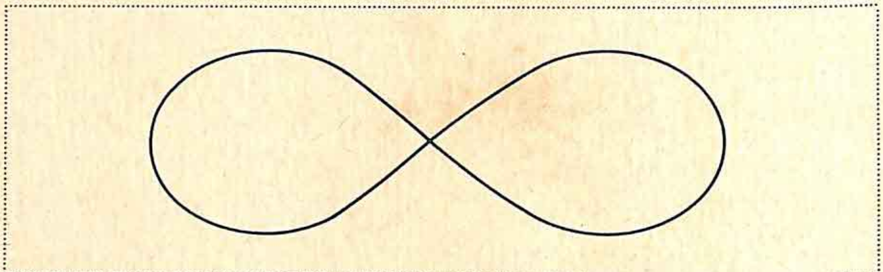
e On the diagram you have just drawn mark a point $x \in$ interior of the curve and a point $y \in$ exterior of the curve.

f If you join the points *x* and *y* by a line how many times does it intersect the simple closed curve?

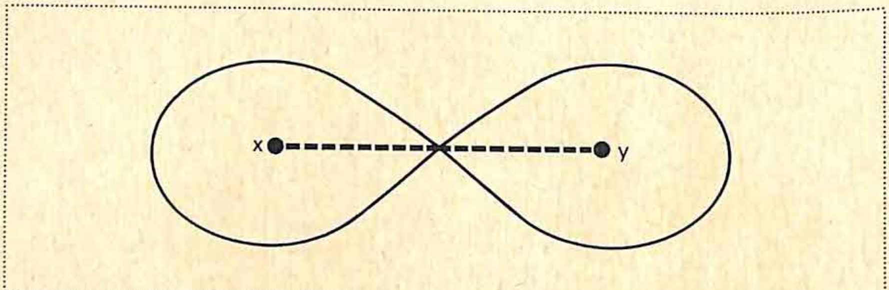
g Can you construct a line joining *x* and *y* and of any shape at all which cuts the simple closed curve an even number of times?

h Suppose you deform the curve like the elastic band of an earlier section. Does what you have just discovered still hold true?

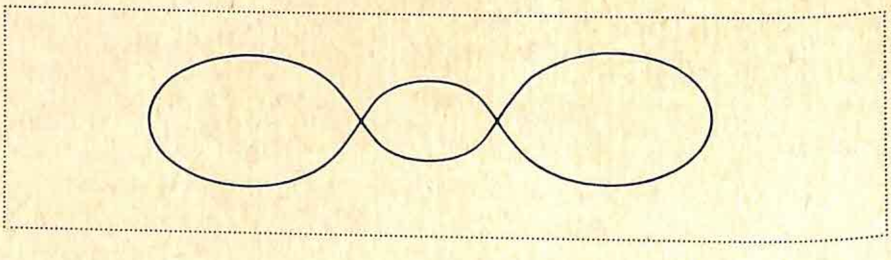
Exercise 3 **a** Here is a curve which is not a simple closed curve.



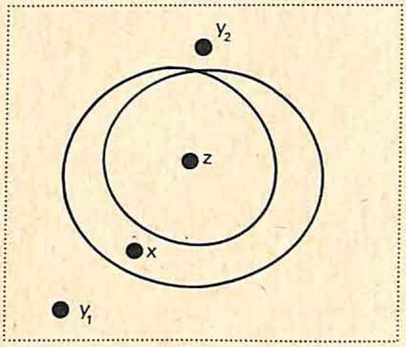
- i What would you mean by 'the interior of the curve'?
- ii Does the rule hold that a line can pass from the interior to the exterior at least once and always an odd number of times?
- iii Can you draw a line which passes from the interior and back to the interior, cutting the curve twice?
- iv Does the rule for an even number of cuttings hold?
- v Suppose we start in the exterior of the curve and cut an even number of times. In which region do we finish?
- vi Explain this drawing:



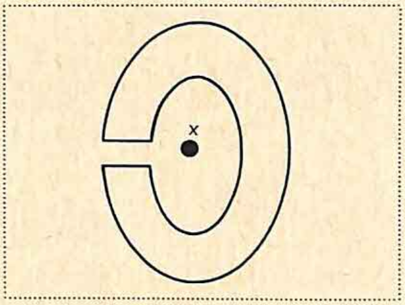
b i Copy this sketch and shade the interior region(s).
ii Do the odd and even rules still hold?



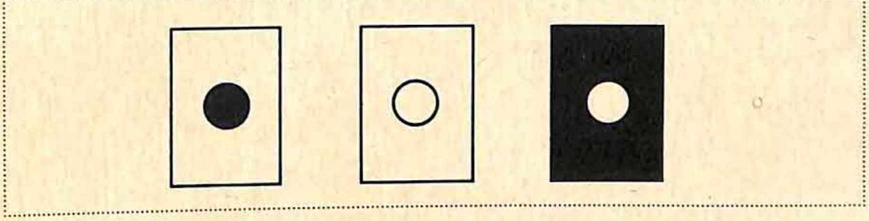
c In this diagram y_1 and y_2 are clearly outside the curve. Use the evens and odds rule to discover what regions x and z are in. Does the result fit with common sense? Do you think that common sense is always to be relied on? Does the line y_2z contradict what the line y_1z tells you?



d Is x inside or outside this curve?

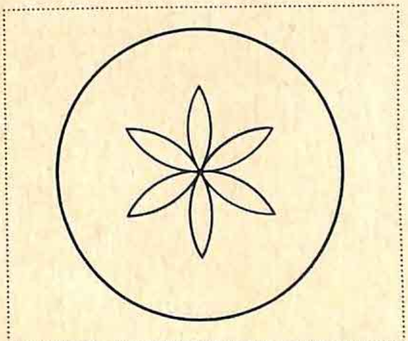


Exercise 4 **a**

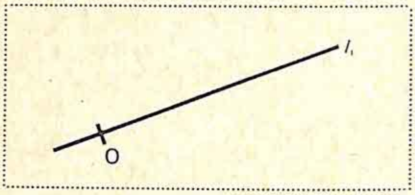


Give a possible meaning for these three diagrams of a circle.

b A round brooch is made of steel into which is inlaid a gold design as in the figure. Into how many regions is the brooch divided? Is the inlay one line-set only?



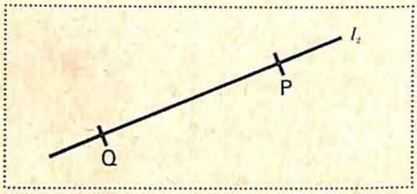
Exercise 5 **a** A point O is an element of a line l_1. What are the three sets illustrated in the diagram?



b Two points P and Q are elements of a line l_2.

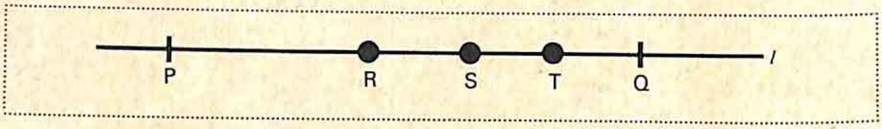
i How many sets are illustrated in the diagram?

ii What are they?



The set *between* P and Q, not including its end points, is an *open interval*. It is an *open set* of points. If we make the union of {P}, {Q}, and the open set {PQ} we get the *closed set* PQ, which is a *closed interval*. We have previously called the interval a *segment*, which is another name for it.

Exercise 6 **a**



i PQ is an interval of the line l. R is any point in the interval. Can I *always* find a point S between R and Q, that is in the open interval RQ?

ii Can I now *always* find a point T lying in the open interval SQ?

iii Can I go on like this 'for ever'?

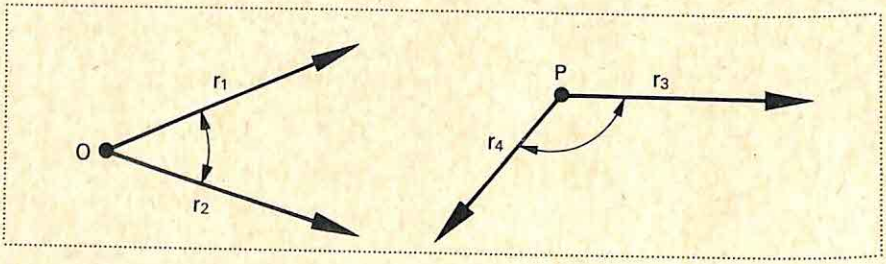
iv Why?

b i The symbol ∃ means *there exists*. Translate this statement into words.

$\forall R \in \{PQ\} : \exists S \in \{RQ\}$

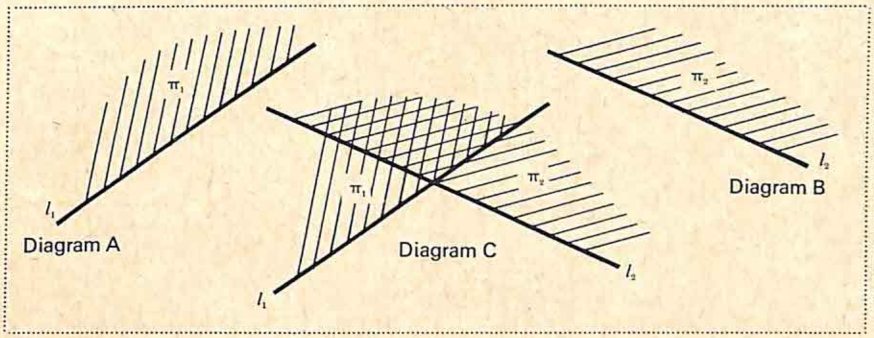
ii With reference to the diagram of 6 a, is this statement true?

We have seen in an earlier section that if two half-lines or rays have the same vertex they are either the same ray or they form an angle.



We can see what an angle is but up to now we cannot say what it is. In the next exercise you will establish a definition of an angle.

Exercise 7 **a** For this exercise you will need a piece of tracing paper in addition to your exercise book.



i Draw a straight line l_1 on your exercise book. Shade one of the half-planes and label it Π_1 (Diagram A).

ii Draw a straight line l_2 in another colour on your tracing paper and shade one of the half planes. Call this half plane Π_2 (Diagram B).

iii Now place the tracing paper on to your book as in Diagram C.

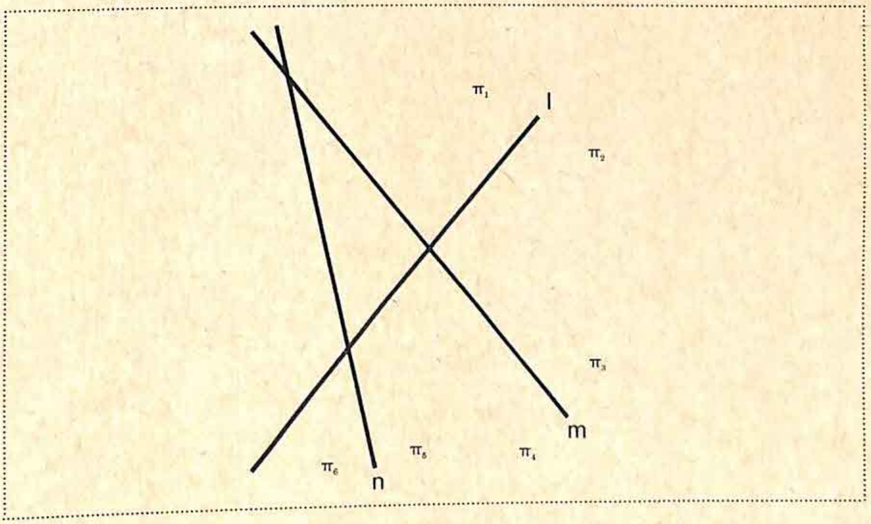
iv The region $\Pi_1 \cap \Pi_2$ is the angle between l_1 and l_2.

v Label two other half-planes on your diagram Π_3 and Π_4.

vi Can you describe another (different) angle between l_1 and l_2 now?

vii How many angles are there between l_1 and l_2? Describe the rest.

b For this exercise you will need two pieces of tracing paper together with your book.



i On your book draw the line *l* and label the two half-planes Π_1 and Π_2. You can colour the two half-planes in different colours if you wish.

ii On the first piece of tracing paper draw the line *m* and label the two half-planes Π_3 and Π_4. Again you may use colours.

iii On the second piece of tracing paper draw the line *n* and label the two half-planes Π_5 and Π_6. Colour them if you want to.

iv Put your three lines on top of one another as in the diagram. Write a description of the region which is the interior of the triangle.

v Write a description of the exterior of the triangle.

Section 10 **Arithmetic with frames and letters**

Exercise 1 Write out the sets:

a $\{x \mid x+4 = 9\}$

b $\{y \mid y-6 = 8\}$

c $\{z \mid 3z = 12\}$

d $\{a \mid 3a+4 = 16\}$

e $\{b \mid 4b-6 = 10\}$

f $\left\{c \,\middle|\, \frac{5c-7}{2} = 14\right\}$

g $\left\{d \,\middle|\, \frac{12d+3}{3}-5 = 20\right\}$

h $\left\{e \,\middle|\, \frac{26-4e}{7}+4 = 2e\right\}$

When you were trying to do that last exercise, you probably found it too hard at some point or other to guess what number would satisfy the condition. You will thus need a method of finding the number. The condition in each case is an *equation* and the number which satisfies it is its solution set or just its *solution.*

We are going to look at the equations without the rest of the statement describing the set. In some cases there will be more than one solution to the equations we are going to write.

The rules for the puzzles you are about to solve are simple. They are these:

1 Each statement is entirely separate from each other statement.

2 Where a box or *frame,* $\square$, $\triangledown$, $\triangle$, $\bigcirc$ appears more than once in a statement it *must* contain the same number each time.

3 Where different shaped frames appear in the same statement, they *need not* contain the same number (but they may do so).

4 The only values we can put into the frames are natural numbers.

Here are some examples:

Solution

A: $\square+4+\square = 8$

$\boxed{2}+4+\boxed{2} = 8$ (2, 2)

	solutions
B: $\square+4+\nabla = 8$	
$\boxed{0}+4+\overset{4}{\nabla} = 8$	(0, 4)
$\boxed{1}+4+\overset{3}{\nabla} = 8$	(1, 3)
$\boxed{2}+4+\overset{2}{\nabla} = 8$	(2, 2)
$\boxed{3}+4+\overset{1}{\nabla} = 8$	(3, 1)
$\boxed{4}+4+\overset{0}{\nabla} = 8$	(4, 0)

	solutions
C: $\triangle+\square+\square = 4$	
$\overset{4}{\triangle}+\boxed{0}+\boxed{0} = 4$	(4, 0, 0)
$\overset{2}{\triangle}+\boxed{1}+\boxed{1} = 4$	(2, 1, 1)
$\overset{0}{\triangle}+\boxed{2}+\boxed{2} = 4$	(0, 2, 2)

	solution
D: $\bigcirc+7 = 11$	
$\textcircled{4}+7 = 11$	4

Where there are two frames in a question, the answer is written as an *ordered pair* or *couple* of numbers. Where there are three frames in a question, the answer is written as an *ordered triple* of numbers. When there is only one frame we shall have only a single number and we shall not bother with brackets for it.

Exercise 2

a
i $3+\square = 7$
ii $7-\triangle = 3$
iii $5+\nabla = 12$
iv $12-\square = 5$
v $7+\square = 14$
vi $14-\triangle = 7$

b
i $4+\nabla = 13$
ii $\nabla+19 = 56$
iii $\triangle+27 = 54$
iv $47+\square = 106$
v $13-\triangle = 4$
vi $\triangle-37 = 19$
vii $54-27 = \nabla$
viii $106-\nabla = 47$

c
i $\square+\square = 6$
ii $\square+\square+\square = 27$
iii $\triangle+\triangle+\triangle = 108$
iv $\nabla+\nabla+\nabla+\nabla = 64$
v $\square+\square+12 = \square+18$
vi $\square+\square+\square+1 = \square+\square+2$
vii $\square+\square+7 = \square+7$
viii $\nabla+7 = \nabla+7$

d
i $\square+\nabla = 6$
ii $\triangle+\triangle+\square = 8$
iii $\triangle+\square = 4$
iv $\square+6 = \triangle$
v $\square+4 = \triangle+4$
vi $\square+4 = \triangle+3$

Exercise 3 **a**

i $\square + \square = 12$
ii $2 \times \square = 16$
iii $\square + \square + \square = 21$
iv $3 \times \bigtriangledown = 63$
v $\bigtriangledown \div 3 = 21$
vi $\triangle \div 7 = 56$
vii $27 \div \bigcirc = 9$
viii $56 \div 14 = \square$
ix $7 \div \square = 7$
x $\triangle \div 1 = 12$
xi $\square + \square = 0$
xii $2 \times \square = 0$
xiii $15 \times \triangle = 0$
xiv $\triangle \times 0 = \bigtriangledown$
xv $\triangle \times \bigtriangledown = 0$

b

i $\triangle + \triangle = 6$
ii $\bigtriangledown + \triangle = 6$
iii $\square + 4 = \triangle$
iv $\square + \bigtriangledown = 0$
v $\triangle + \triangle = \triangle + 2$
vi $\triangle + \square = \triangle + 2$
vii $\triangle + \square = \square + \triangle$
viii $\triangle \times \bigtriangledown = 12$
ix $\triangle + \bigtriangledown = 13$
x $\square + \triangle = \triangle$
xi $\square \times \triangle = \triangle$
xii $\triangle \times \triangle = \triangle$

c

i $\triangle + \bigtriangledown + \triangle = \triangle + 2 + 3$
ii $\triangle + \bigtriangledown + \triangle = \triangle + \bigtriangledown + 3$
iii $\triangle + \bigtriangledown + \square = \square + \triangle + \bigtriangledown$
iv $\triangle \times \bigtriangledown \times \square = \triangle \times 3 \times 2$
v $\triangle \times \square \times \triangle = \triangle \times 3 \times 2$
vi $\triangle \times \square \times \triangle = \square \times 3 \times 3$

d

i $\triangle \times (3+2) = (\triangle \times 3) + (5 \times 2)$
ii $\triangle \times (3+2) = (\triangle \times 3) + (\triangle \times 2)$
iii $\square \times (5+4) = \square \times 5 + \square \times 4$
iv $\bigtriangledown(7+8) = 7\bigtriangledown + 8\bigtriangledown$
v $8(4 + \bigtriangledown) = 8 \times 4 + 8\bigtriangledown$
vi $8(3 + \triangle) = 24 + 40$
vii $8(4 + \triangle) = 32$
viii $8(\triangle + \bigtriangledown) = 48 + 40$
ix $8(\triangle + \triangle) = 88$
x $4(\bigtriangledown + \triangle) = 40$
xi $\bigtriangledown(\bigtriangledown + \triangle) = 17$
xii $\bigtriangledown(\square + \triangle) = 17$

We have been using frames up to now, but we could just as well use letters of the alphabet. We always use small letters for this purpose. Instead of $9 + \square = 12$, we could write $9 + x = 12$. The question looks a little different, but the idea is the same. We are still looking for the member of the set of natural numbers which makes a true statement when we put it in place of x. When we do this we may write the solution as $x = 3$.

Thus we have two ways of writing our question and answer:

A: $\{x \mid 9 + x = 12\} = \{3\}$

or

B: $9 + x = 12 \Leftrightarrow x = 3$

The sign $\Leftrightarrow$ means that the two statements it separates are equivalent to each other, that they both tell us the same thing.

Whether we use method A or method B will depend on how the question is asked. If it is asked in set-language, then we reply in the same language. But if the shortened form is used, we answer in the short way.

Here is another example in the shortened form:

$y + 4 = 17 \Leftrightarrow y = 13$

Exercise 4 Use the short form to give the solutions of the following.

a $a+20 = 24$

b $20-b = 14$

c $6-c = 2$

d $12-\mathrm{b} = 0$

e $8+d = 16$

f $4+d+d = 14$

Before we can go any further we shall need another short form. Let us look again at multiplication and addition.

Exercise 5 **a** Work out

i $7+7+7+7$

ii 4×7

iii $9+9+9$

iv 3×9

v $5+5+5+5+5+5$

vi 6×5

b Use this principle to shorten the following. The answers are at the end of the section.

i $a+a+a+a$

ii $b+b+b+b$

iii $c+c+c+c+c$

iv $x+x+x$

v $y+y+y+y+y+y+y$

vi $z+z+z+z$

c If we leave out the × sign between a number and a letter we shall not cause any confusion. $4\times y$ can be written $4y$. Shorten the six answers in (b) in this way.

d We can also agree to leave out the 1 in $1x$, writing just x. After all we have been doing this already and have understood it. Now shorten the following expressions.

i $x+4x$

ii $y+3y$

iii $4z+z$

iv $5a+2a$

v $1b+3b$

vi $3c-c$

vii $5d-2d$

viii $7e-4e$

e Shorten these expressions as much as you can. You may check alternate answers at the end of the section.

i $x+x+x+5$

ii $y+y+y+y+7$

iii $z+z+8+z$

iv $p+p+5-p$

v $q-q+q-2$

vi $9+r-5-r$

vii $8-12+r+14$

viii $17-s-s+3+s$

ix $2a+5-a+3a$

x $6b-7+2b-2$

xi $7c-9c+2+3$

xii $8d+4-8+2d$

Exercise 6 **a** Find solutions of the following:

i $2x = 12$

ii $7y = 35$

iii $24y = 48$

iv $17y = 119$

v $56z = 112$

vi $21a = 105$

b i $3x+4 = 10$
ii $4y+7 = 19$
iii $7z-4 = 3$
iv $3z-2 = 2z$
v $4z+1-3z = 5$
vi $3a+a-4 = 2$
vii $b-6+3b = 6$
viii $4+c-6+c+2 = 0$

Answers to Exercise 5

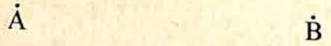
b i $4 \times a$
ii $4 \times b$
iii $5 \times c$
iv $3 \times x$
v $7 \times y$
vi $4 \times z$

e i $3x+5$
iii $3z+8$
v $q-2$
vii $10+r$
ix $4a+5$
xi $5-2c$

Section 11 **Moving in circles**

Exercise 1 **a** If a beetle moves from one spot to another, how can we find out how far the beetle has moved?

b



Suppose the beetle moves from point A to point B on this diagram. How far has he gone?

c Is there only one answer to the last question? Is there only one distance from A to B?

d What is the shortest distance from A to B?

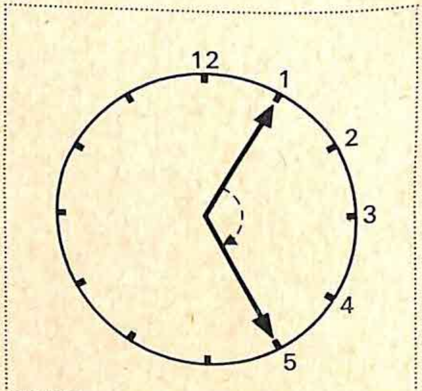
e Along what path would he have moved to cover the shortest distance? We call the effective distance he has moved (the shortest distance) his *displacement* and we need a ruler to measure it.

Exercise 2 **a** If the minute hand of a clock moves from '5 past' to '25 past', how far has it moved? Can you use a ruler to find the displacement?

b Is the movement the same on Big Ben as on a wrist watch?

c What is it we are trying to measure?

d Does *distance* have any meaning here?



Clearly we are trying to measure not length or distance, but an amount by which the hand turns, an *amount of turning*. This is the *measure of the angle* through which the hand has turned. Let us find out how to measure angles.

Exercise 3 **a** The hand of a clock moves from '5 past' right round the clock once to '5 past' again. How would you describe the movement?

b If it began at 'a quarter past' and completed one rotation, where would it end?

c If it began at '10 to' and completed one turn, where would it end?

d If it began at '25 to' and completed a half-turn, where would it stop?

e If it began at '19 minutes past' and completed a half-turn, where would it stop?

f If it began at '3 minutes past' and completed a quarter-turn, where would it stop?

g If it did a quarter-turn from '12 minutes to' where would it stop?

h Complete the table:

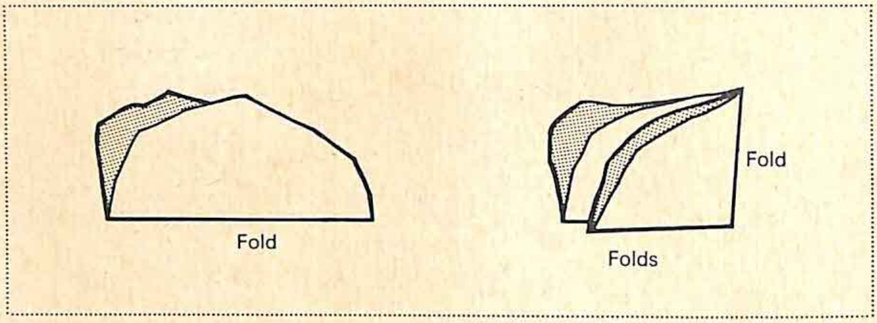
	Begin	Turn	End
i	5 to	$\frac{1}{4}$	
ii	22 to	$\frac{1}{2}$	
iii	22 past		8 to
iv		$\frac{1}{4}$	23 to
v	15 to	$\frac{3}{4}$	
vi		$\frac{3}{4}$	15 to
vii	35 past	$\frac{1}{4}$	
viii		$\frac{1}{2}$	45 past
ix	2 past	$\frac{3}{4}$	
x	32 past		17 past
xi		$\frac{1}{8}$	22 to
xii		$\frac{1}{12}$	17 past

The clock face is divided into smaller sections than half- and quarter-turns. It is clearly marked in twelfths and less clearly in sixtieths of a turn.

Exercise 4 What period of time corresponds to the following fractions of a turn (of the minute hand)?

a $\frac{1}{60}$
b $\frac{1}{12}$
c $\frac{7}{60}$
d $\frac{1}{6}$
e $\frac{7}{12}$
f $\frac{1}{3}$
g $\frac{5}{6}$
h $\frac{2}{3}$

Exercise 5 **a** Take a sheet of writing paper and make in it a single fold. Then fold it again so that the fold is bent back on itself.



b The corner of the paper is now a *right-angle*. What fraction of a turn corresponds to one right-angle?

c How many right-angles correspond to a three-quarter-turn?

We can divide the circle as we please into suitable units. The size of the circle does not matter.

The Babylonians about 4000 BC were sure there were 360 days in a year. So that they could track the sun to work out their calendar, they divided the circle into 360 units. We still do this and we call the units *degrees*. A half-turn is thus 180 degrees or 180°.

Exercise 6 **a** How many degrees are there in

i 1 right-angle
ii 3 right-angles
iii 4 right-angles
iv 2 right-angles?

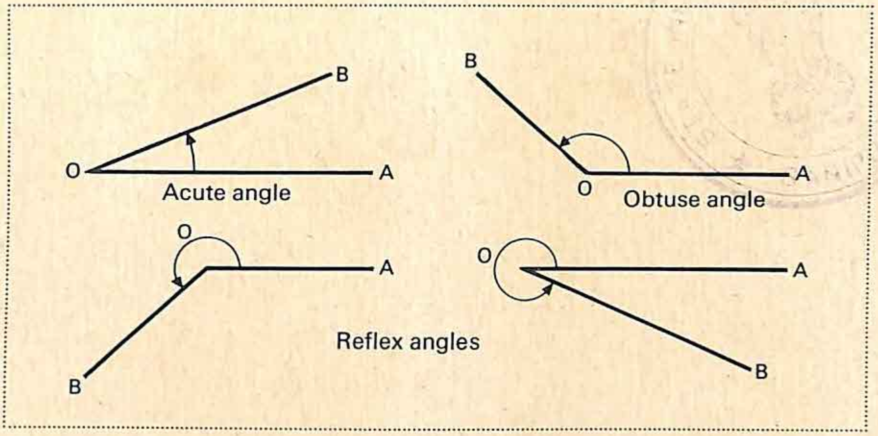
b How many degress are there in

i $\frac{1}{2}$ right-angle
ii $\frac{3}{4}$ right-angle
iii 5 minutes on the clock face
iv 1 minute on the clock face
v 50 minutes on the clock face
vi 25 minutes on the clock face?

c What are the angles (measured clockwise) between the two hands of a clock at the following times? Give for each the answer in right-angles, and degrees.

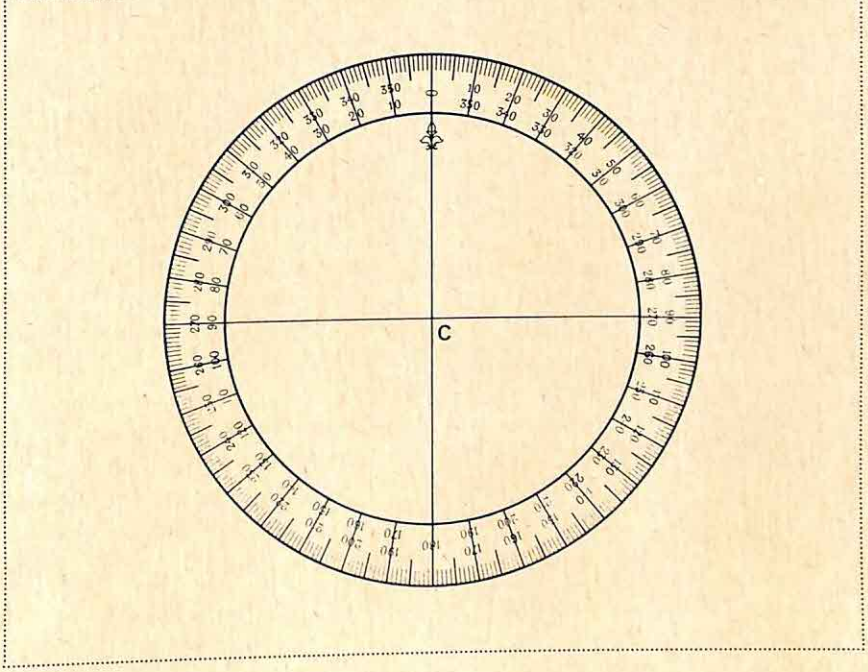
i 1 o'clock
ii 5 o'clock
iii 3 o'clock
iv 8 o'clock
v 9 o'clock
vi 10 o'clock

There are three special names we need to know:
If an angle is smaller than 1 right-angle, it is called an ACUTE angle.
If an angle is greater than 1 right-angle and less than 2 right-angles, it is called an OBTUSE angle.
If an angle is greater than 2 right-angles and less than 4 right-angles, it is called a REFLEX angle.

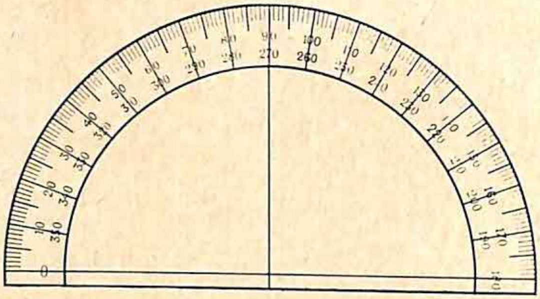


Exercise 7 Say whether each of the angles of Exercise 6(c) measured clockwise are acute, obtuse or reflex.

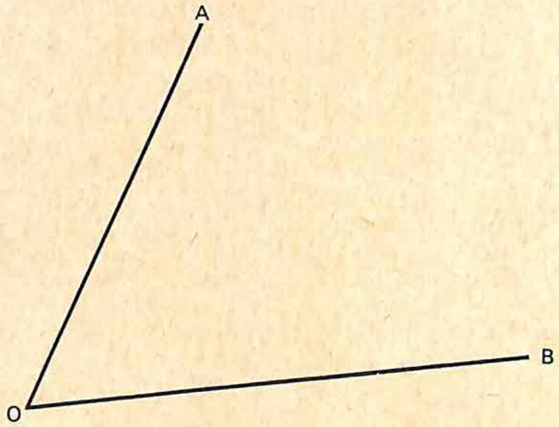
An instrument for measuring angles is the *protractor*. They may look like either of our two illustrations. Notice that the semi-circular one has a protective edge to it. You have to ignore this part when you are using the protractor.



Which way do the numbers go?

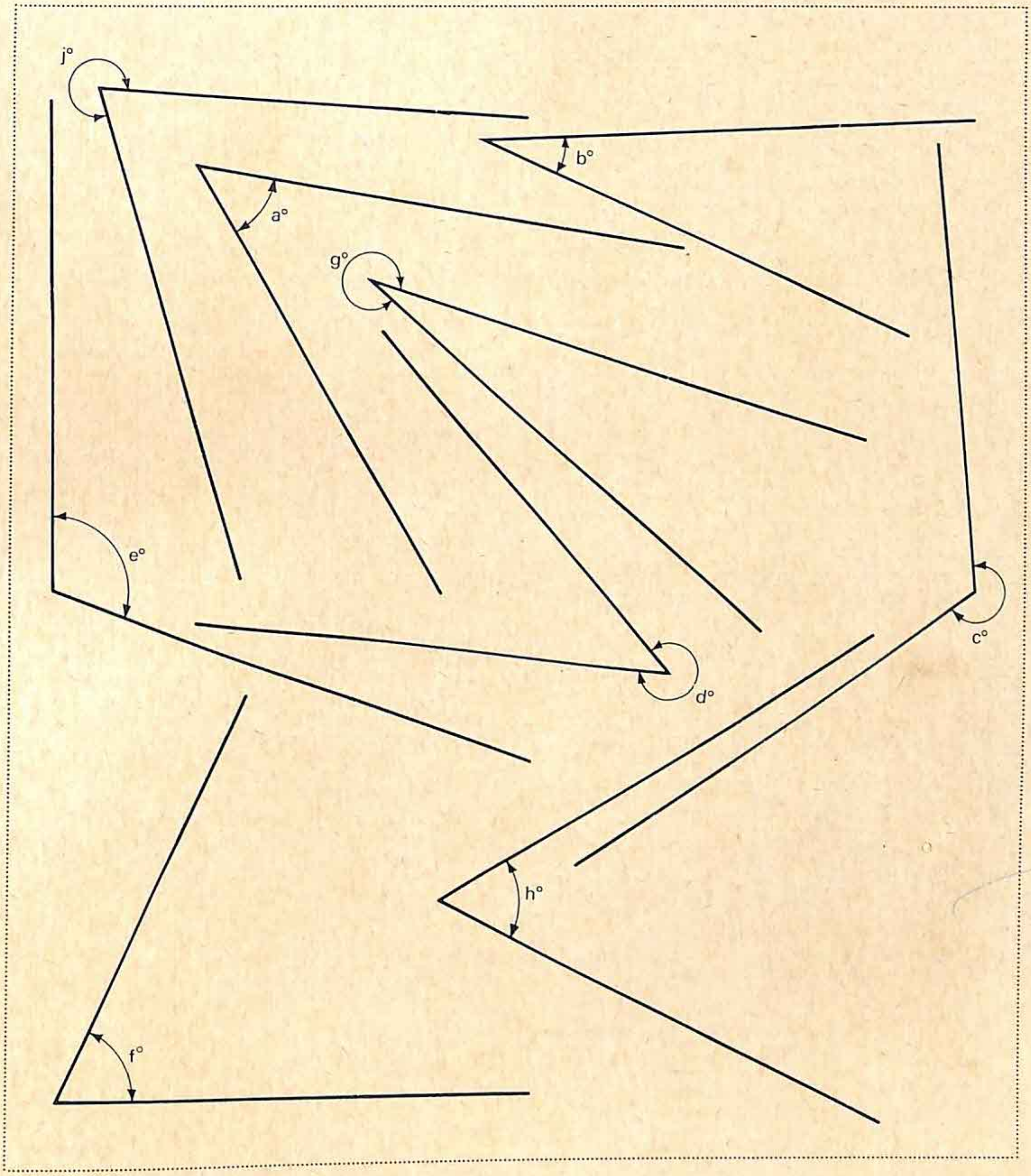


We want to draw an angle of 58°. We want to call it angle AOB. It will look something like this:



First we decide whether the angle we are trying to draw is acute or obtuse: in this example acute. We draw the line segment OB any length. We place the point C of the protractor over the vertex O and its base line (0°) along the line OB. We then put a dot against the number 58 which will make AOB an acute angle. If we choose the wrong one we shall get the obtuse angle 122°. We can measure an angle in exactly the same fashion.

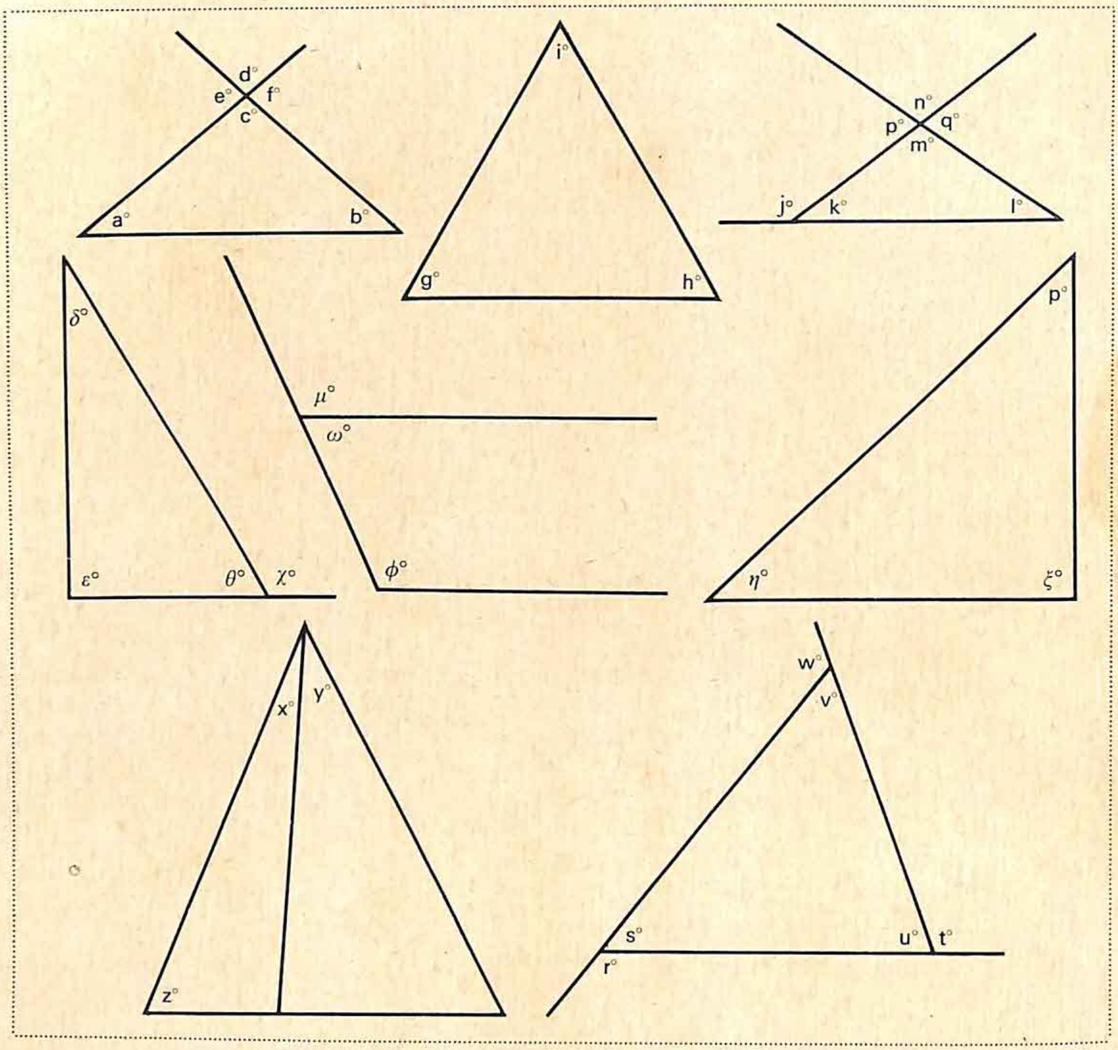
Exercise 8 Measure these angles carefully:



Exercise 9 Draw the following angles carefully:

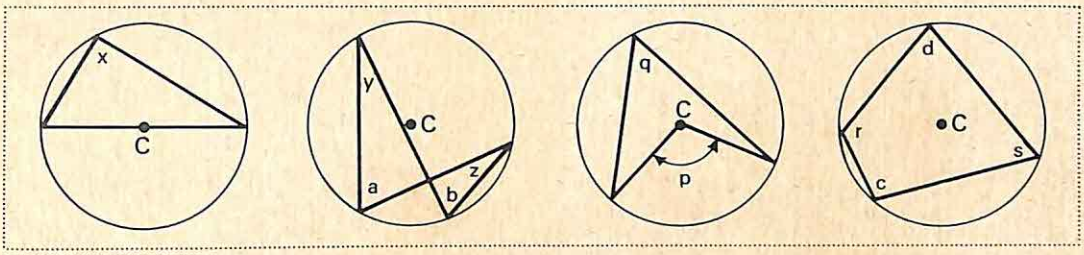
a	27°	**c**	270°	**e**	305°	**g**	125°
b	132°	**d**	224°	**f**	156°	**h**	354°

Exercise 10 Measure all the angles on these diagrams.



As you saw, in the last exercise we ran out of English letters so we had to turn to the Greek alphabet for more. Mathematicians often have to do this. In fact there are as many as six alphabets in use by mathematicians to say nothing of all the special signs. You will not be meeting them all in this book!

Exercise 11 Copy these diagrams on a larger scale. They need not be exactly the same as the ones in the book, in fact you should do three different ones of each. The only things you must be sure about are that a line going through C (the centre of the circle) in the book does so in your copies, and that points on the circle in the book are also on the circle in your copies.



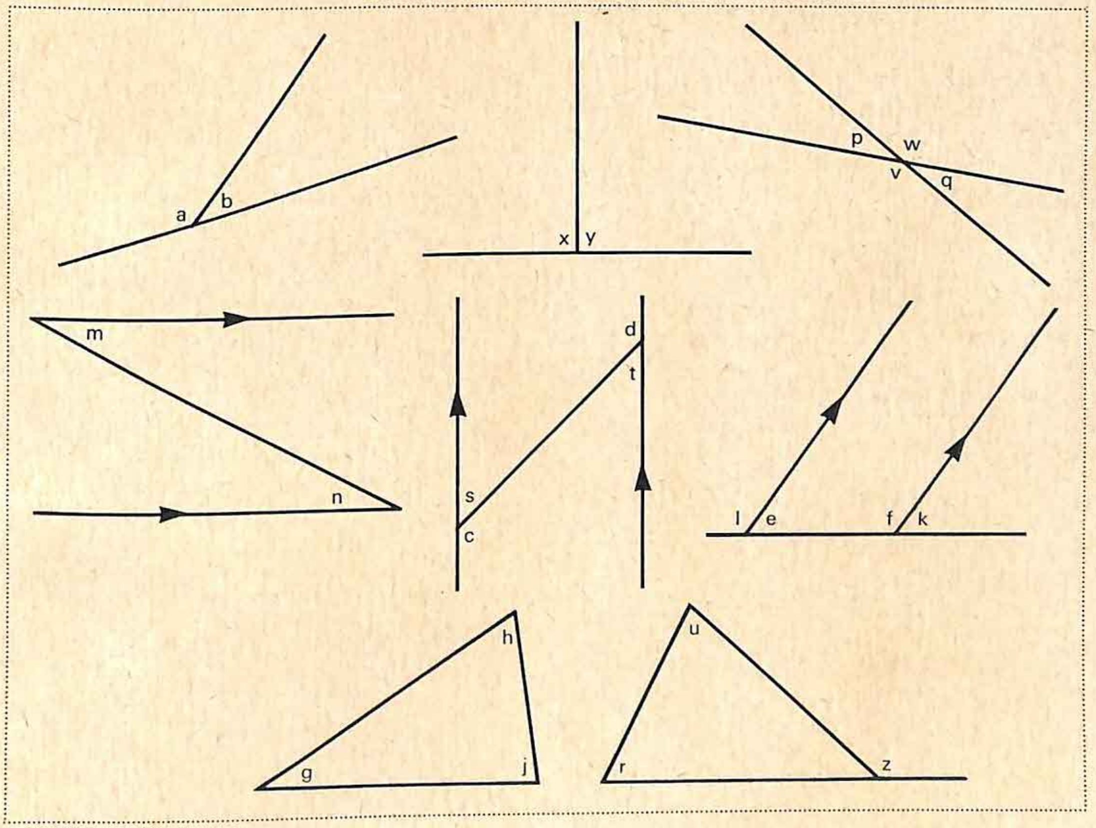
a Measure all the angles in the diagrams and write them on your drawings.

b These are special angles to notice:

i x
ii y, z
iii a, b
iv p, q
v r, s
vi c, d

Exercise 12 Here are some more sketches for you to copy. Again you need not be sure of exact copies and you should make at least three different ones of each.

a When you have copied them again measure all the angles. (The arrows mean *parallel* lines.)

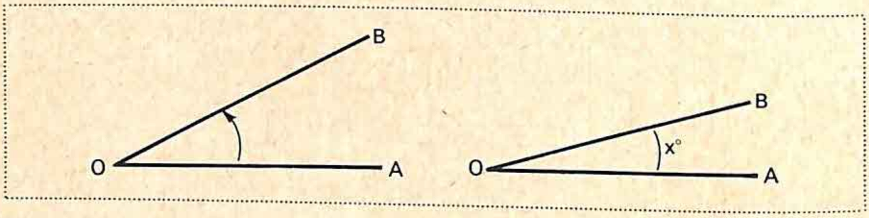


b Here are some special angles to notice

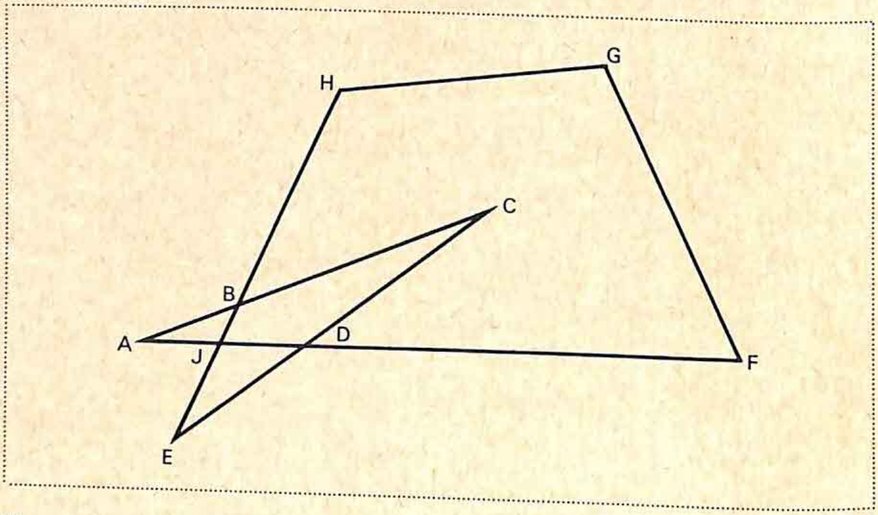
i a, b
ii x, y
iii p, q
iv v, w
v m, n
vi s, t
vii c, d
viii e, f
ix g, h, j
x r, u, z

c You may have reached certain conclusions about certain kinds of angles from this exercise and the last. Can you at present be sure of these results? You got them by drawing remember.

We can name an angle in three ways. We can either: (a) call it 'the angle at O' and write down '$\hat{O}$'; or (b) call it either 'the angle AOB' or 'the angle BOA' and write '$A\hat{O}B$' or '$B\hat{O}A$'; or (c) we can write inside the angle a small letter or sign and use this to represent the size of the angle.



Exercise 13 Copy this figure.



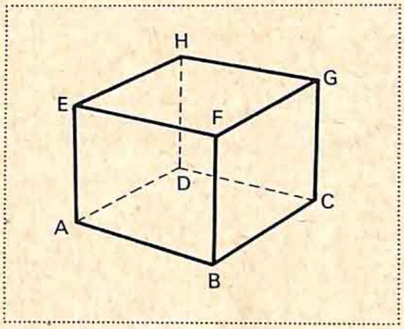
To mark angles in a figure you can use small letters. Below you are given the three-letter names of four angles and a one-letter name for one angle. Mark each angle on your copy with the small letter given in brackets.

a $A\hat{B}H$ (v)
b $\hat{C}$ (w)
c $D\hat{F}G$ (x)
d $G\hat{H}B$ (y)
e $E\hat{B}C$ (z)
f Could you mark $\hat{J}$?

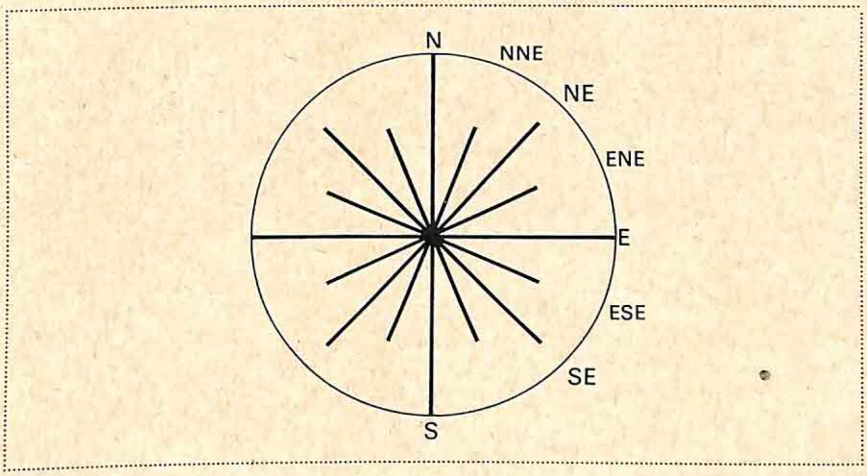
Exercise 14 The drawing represents a block of wood. There are no right-angles in the drawing but there are twenty-four on the block of wood.

a To name an angle you will have to use three letters. Why will one letter not do?

b Name all the twenty-four right-angles.



Exercise 15 Measure the angles of your set-squares.



There are several ways you could give compass bearings. In any case you will use degrees. You might say N 5° E means 'a direction clockwise 5° from the direction of north'; W 20° S would then mean 'anticlockwise 20° from west'. The difficulty is that bearings might have several different names.

Exercise 16 a Say which of the following are the same bearings:

i E 25° S
ii S 25° E
iii W 25° S
iv S 65° E
v E 65° S
vi S 65° W
vii E $22\frac{1}{2}$° S
viii S $22\frac{1}{2}$° E

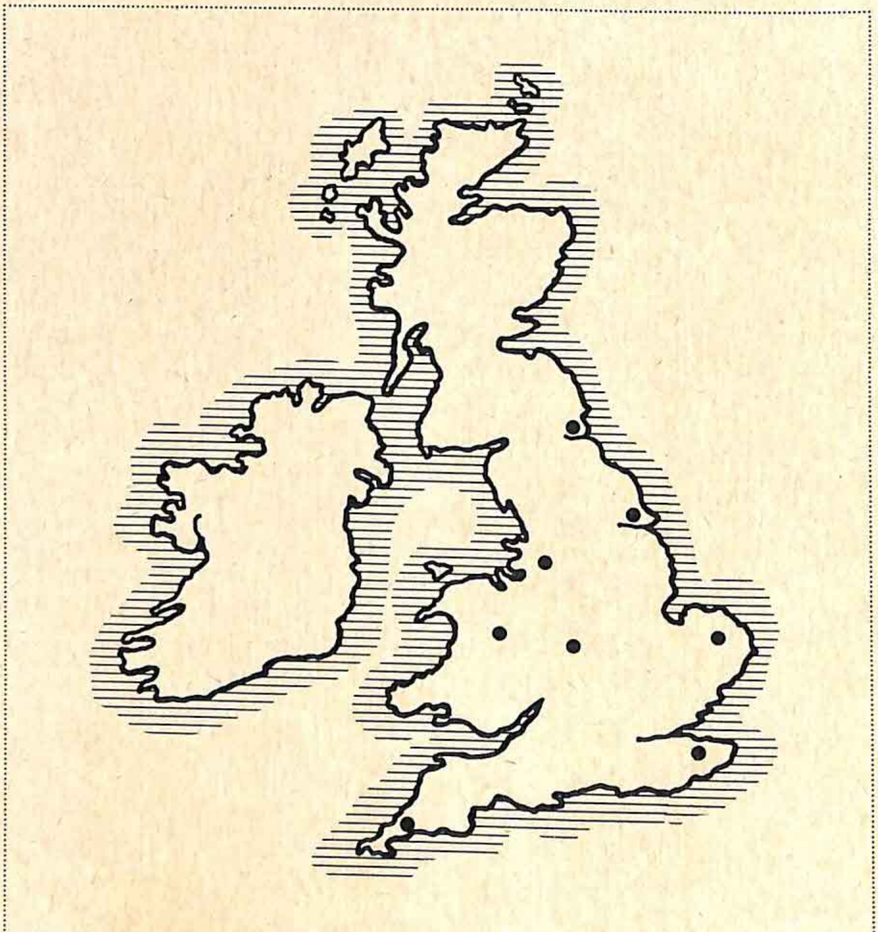
b Can you find a meaning for N 135° S? Is there only one meaning for it?

There is a much better method of giving bearings provided we all agree on how to write and interpret them. First, we always use a three digit number; second, we always read clockwise; third we always start from N. A bearing of 090° is thus due E, and 135° is SE. 225° is then SW.

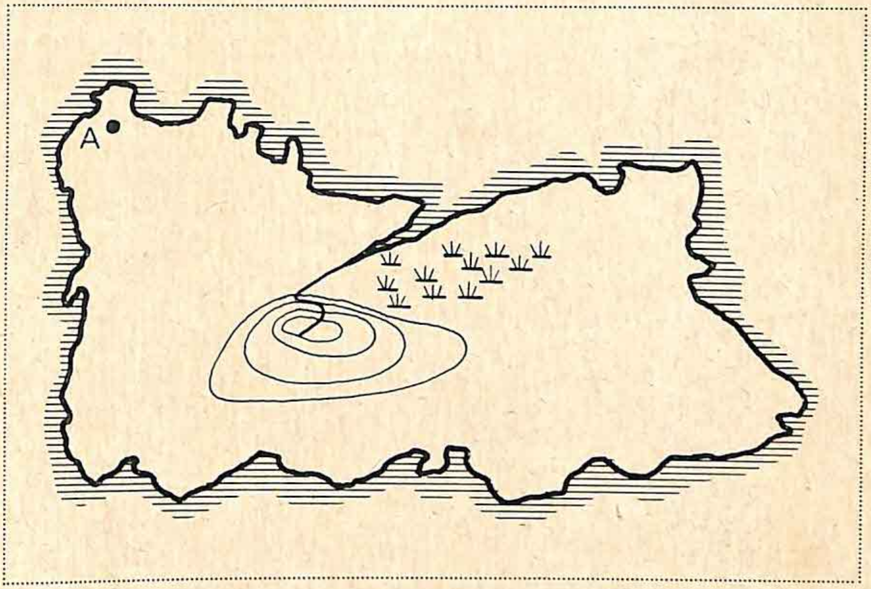
Exercise 17 a Write these bearings using this convention.

i NW
ii NE
iii SSE
iv ESE
v WSW

b On an outline map of England mark your home town and also Manchester, Birmingham, Newcastle upon Tyne, Kingston-upon-Hull, Norwich, Canterbury, Shrewsbury, Truro. Draw lines to each of these from your town and work out the bearings of all the others by using your protractor.



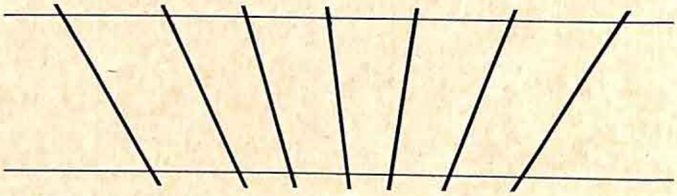
c This map shows a treasure island. It is drawn so that 1 in. represents 2 miles. Trace the map and plot the following on your tracing.



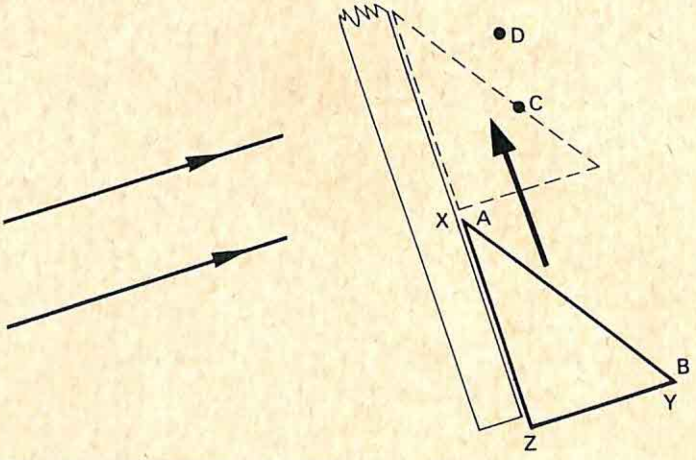
From landing place A go 3 miles due South to B; then go to C in a straight line 6 miles on a bearing of 067°; next, change direction clockwise through an angle of 83° and travel straight for 4400 yds. to D. The treasure is at E in a direction W 10° S and at a distance of $2\frac{3}{4}$ miles. How far and on what bearing is the treasure from A?

Section 12 **About parallels**

Draw along both long edges of your ruler. Now draw a few lines across the figure sloping various ways. Measure the segments of the crossing lines (or *transversals*) which lie between the lines. Are they all the same length? Which is the shortest transversal you can draw? If you want to talk about the distance between the lines, which measurement will you give? Measure the distance at various places along your two original lines. What do you discover? Straight lines in a plane which are the same distance apart all along their length are called *parallel* lines. As we have seen before, parallels 'go in the same direction'.



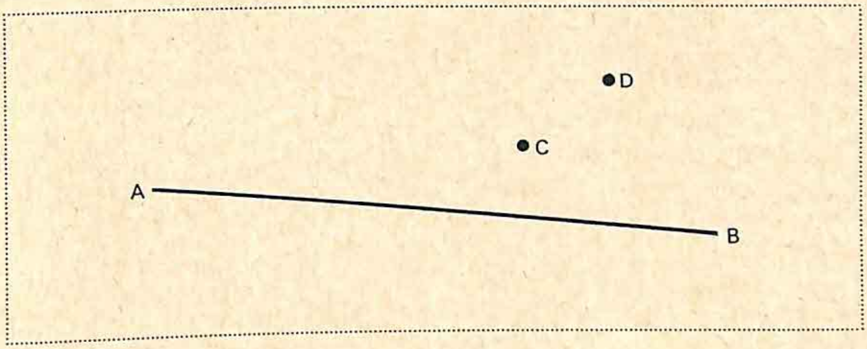
Can you find some parallel lines in the room you are in? Look at your exercise book. Can you see a set of parallel lines? (That is, a lot of lines all parallel to each other.) A set of parallel lines is sometimes called *a*



family of parallel lines. We have a way of indicating that lines on a figure are parallel by placing small arrows on them. Do parallel lines ever cross one another? Why not?

Suppose we want to draw two lines through points C and D and parallel to line AB. Here is one method of doing it. Place one edge XY of any set square along AB. Now place your ruler along one of the other edges of the set square, and hold it very firmly. Now slide the set square carefully along the ruler until XY passes through C. Draw along XY. Repeat for point D.

Exercise 1 **a** Draw a line AB and two points C and D not on the line.



Use the set-square method to draw parallels to AB through C and D.

b Repeat (a) with another line and two more points, this time using your set-square another way round.

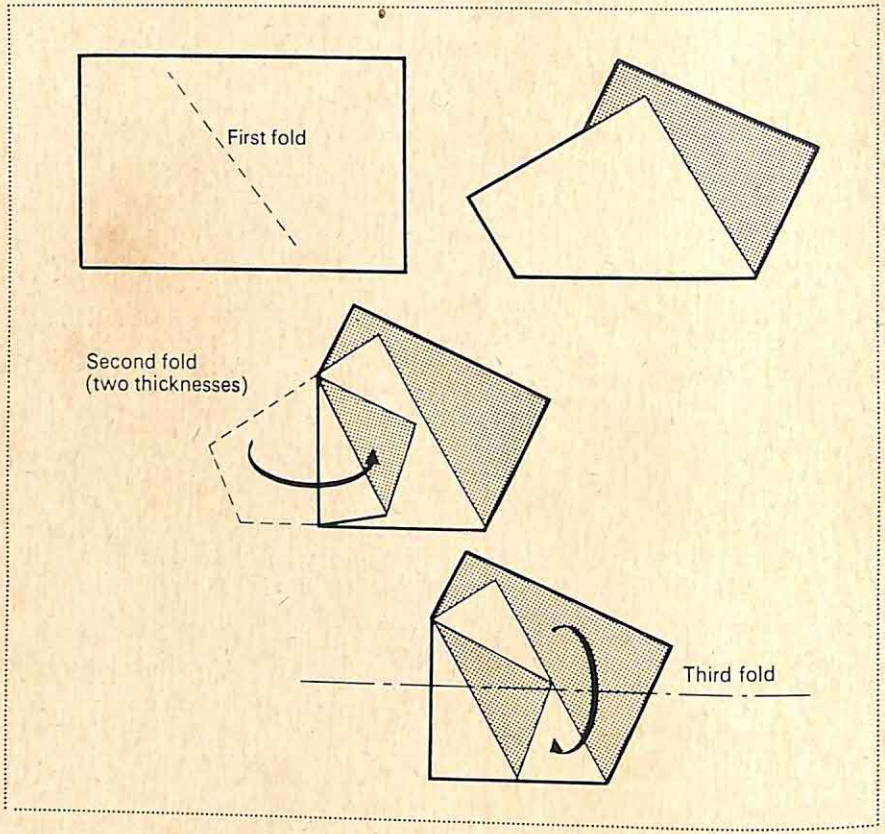
c Do the same thing again using your other set-square and drawing the line through D with the set-square a different way round from the way you did the line through C.

Exercise 2 Here is a paper folding method of 'drawing' parallels.

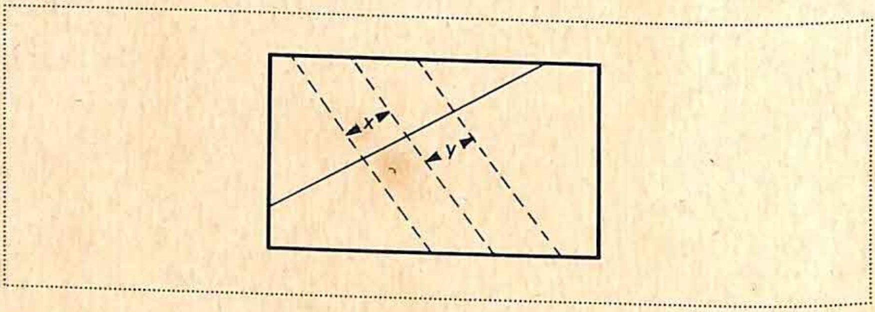
Take a sheet of paper and somewhere near the middle fold a single sharp crease.

Now fold the crease back on itself so that you get a right-angle corner.

One edge is now a single fold with two thicknesses of paper, while the other is two separate folds of a single thickness each. Fold the two-thickness fold back on itself. Open out. You have three parallel folds and a transversal.



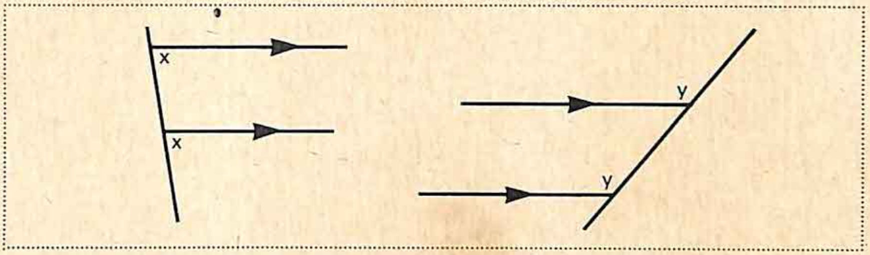
a What can you say about the distances x and y?



b What can you say about all the angles in the figure?

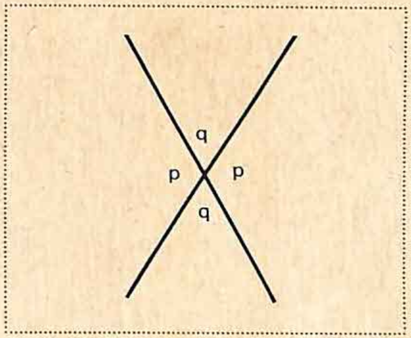
Save this sheet of paper for Exercise 4.

Exercise 3 **a** If three lines make a shape like a letter F (maybe upside-down or backwards) the two angles marked x are *corresponding* angles and are equal.



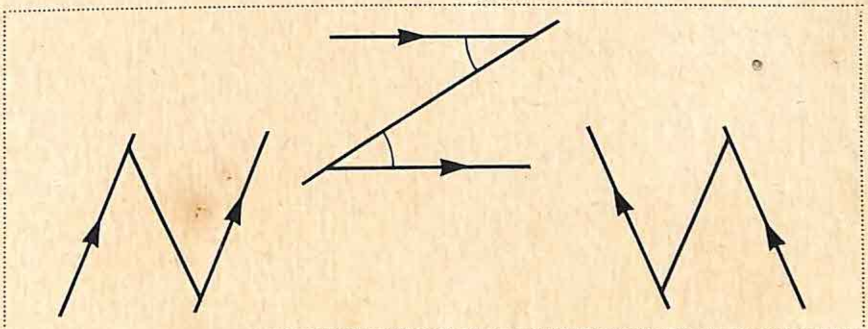
i What would you say about the two angles marked y?
ii Can you find some *corresponding angles* in your figure from Exercise 2? Make a list of four pairs.

b If two lines form an X there are two pairs of equal angles called *vertically opposite* angles. One pair is marked p in the diagram.



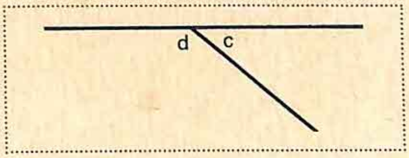
i What can you say about the angles marked q?
ii List four pairs in your diagram of Exercise 2.

c If three lines form a Z or an N (maybe backwards) there are two equal angles called *alternate angles.*



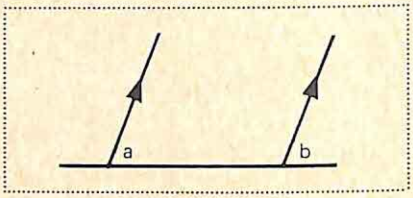
i Copy the second two diagrams and mark the alternate angles.
ii Find two pairs of alternate angles in your diagram of Exercise 2.

d If two lines form a T or if three lines form two Ts, there are some special angles. They are named in the diagrams. These angles are not equal.



i What do you know about the angles marked *a* and *b* in the diagram?

ii Is the same thing true of *c* and *d*?



iii *Supplementary* is a single word which means 'adding up to 180°'. Can you list ten pairs of supplementary angles in your figure of Exercise 2?

Exercise 4 Take your folded paper from Exercise 2 and fold a new transversal, not parallel to the one already there.

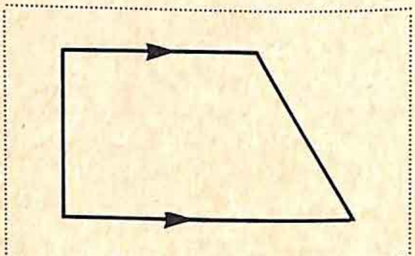
a How does this new transversal differ from the previous one with regard to the angles it forms with the parallels?

b What can you say about the distances between the parallels measured along the new transversal?

c How does that distance differ from the *x* and *y* of Exercise 2?

d If you had made your new transversal parallel to the old instead of not parallel, what sort of four-sided shapes would have been made on the page?

e As it is you have a shape like this:

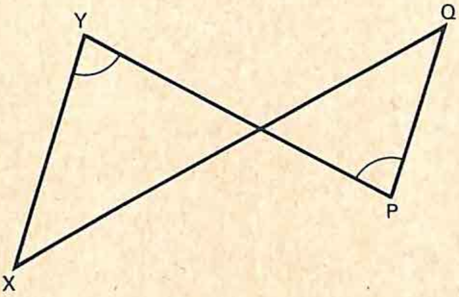


This is a special sort of *trapezium.* Have you any more trapeziums like this? Would you have if all your folds were infinite lines?

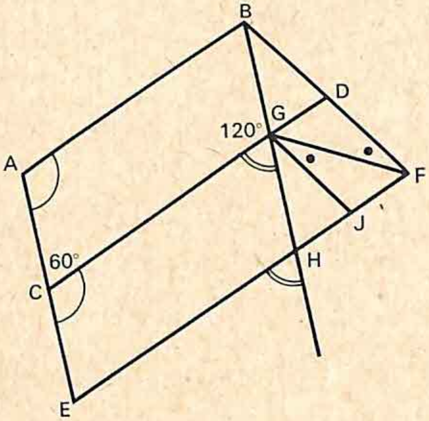
Exercise 5 Name all the sets of parallel lines in the following figures. Say why each pair of lines is parallel and do not assume they are because they look as though they are. You can most easily give your answers in a table like this:

line	is parallel to line	because angle	equals angle	type
XY	PQ	XYP	YPQ	alternate

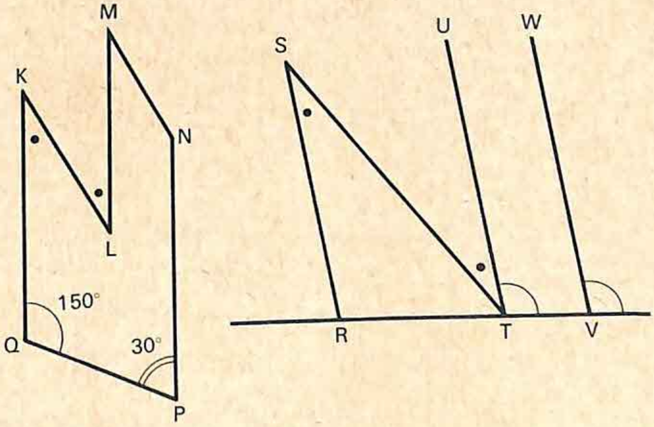
That would be the answer for this diagram:



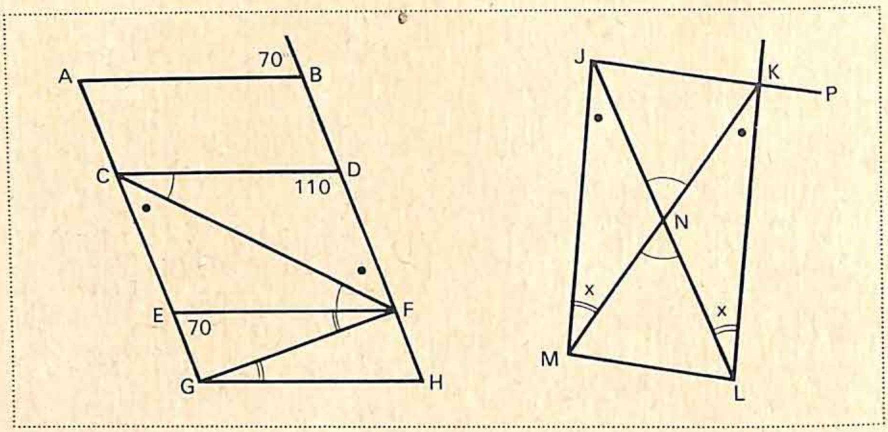
a



b, c



d, e



f Explain why in (e) angle LKP > x.

Exercise 6 **a** In how many directions at right angles to each other can you draw lines on a flat sheet of paper?

b In how many directions at right angles to each other can you see lines in a room?

c If you have a cube, how many directions at right angles can you find?

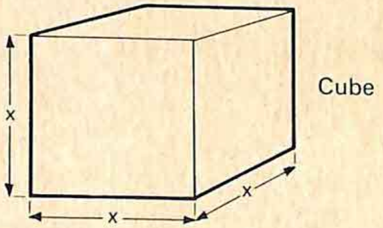
d The word for 'directions at right angles' is *dimensions*. A flat surface is 2-dimensional; a room or a wooden block is 3-dimensional. What sort of thing is 1-dimensional?

e Can you imagine anything which is 4-dimensional?

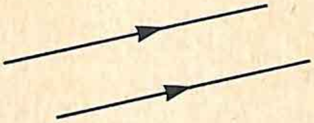
f Is a piece of hardboard 1-, 2- or 3-dimensional?

g Is a penny 1-, 2- or 3-dimensional?

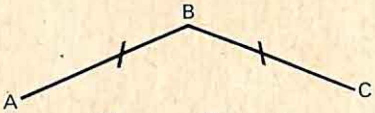
h How many dimensions has a sheet of paper?



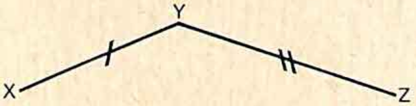
In the following exercise we shall mark some lines as parallel with small arrows.



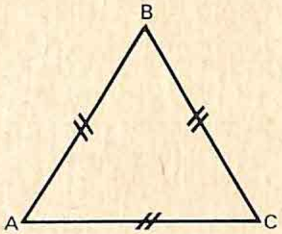
Equal line-segments will be marked with short dashes.



The two lines are
equal, AB = BC



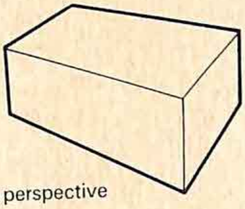
These lines are *not*
equal, XY ≠ YZ



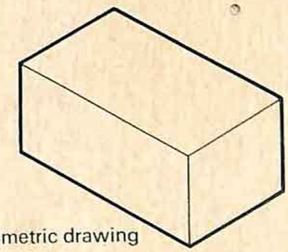
These three lines are
equal, AB = BC = CA

When we want to draw a 3-dimensional object, we are trying to represent it on a 2-dimensional surface. There are several ways of doing this.

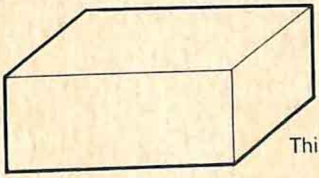
Here is a drawing of a block done in three different ways.



This is in perspective

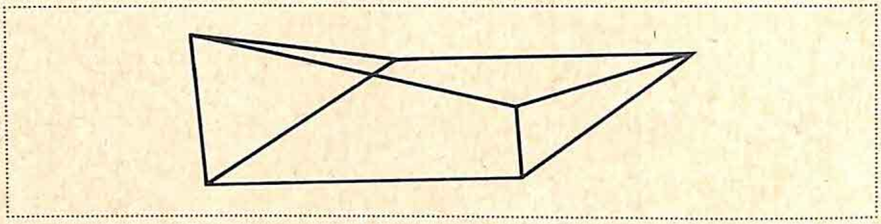


This is an isometric drawing

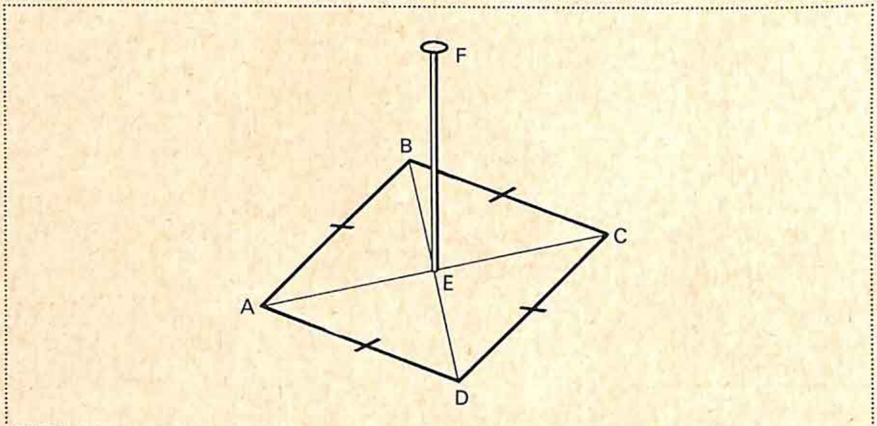


This is an oblique drawing

It does not matter which we use except that we need to be told that our drawing is 2- or 3-dimensional. Which do you think this drawing is?



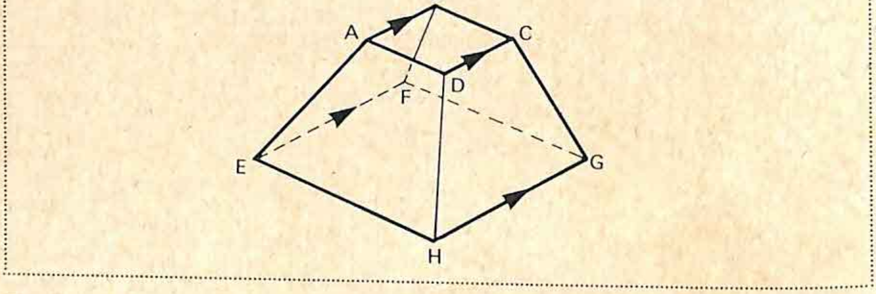
Exercise 7 Here are some drawings. You are told which number of dimensions they represent, and you are given some other information on the drawing.



a A flag pole standing vertically in a square. Three dimensions.

i Say which line segments are equal.

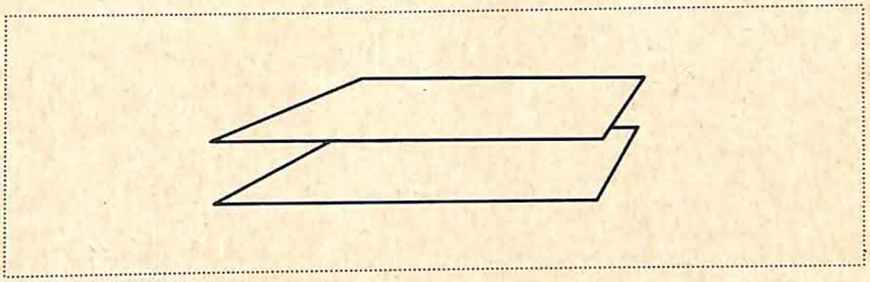
ii Say which line segments are parallel.



b Three dimensions. Pyramid with the top missing. Square base.

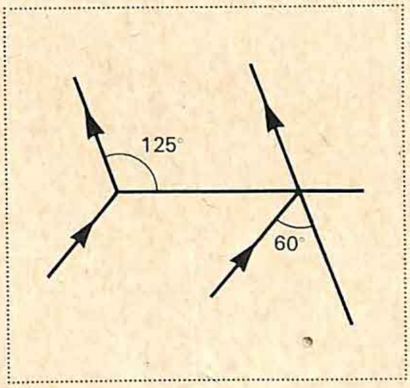
i With the information given, can you say whether AD is parallel to EH?

ii If AD is parallel to EH, what can you say about BC and FG?
iii From the given information say which edges are equal.

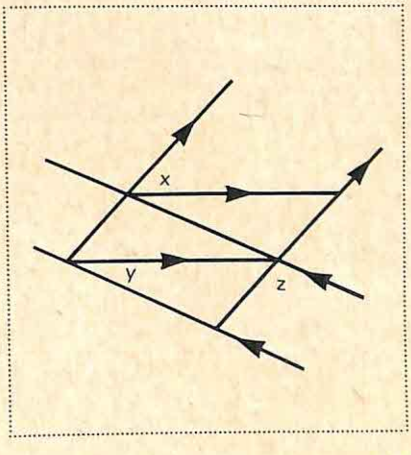


c Three dimensions. Two planes like the biscuits of an ice-cream sandwich.
i Would you describe them as parallel planes?
ii Is any line in one plane parallel to any line in the other?
iii Can two non-parallel planes be arranged so that they do not meet?
iv When two planes do meet, what sort of line is formed?

d Two dimensions
Copy the diagram and find the sizes of all the angles.

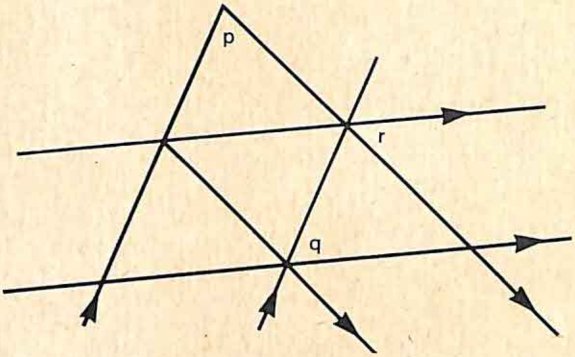


e Two dimensions
Copy the diagram and find the sizes of the angles in terms of x, y, z.



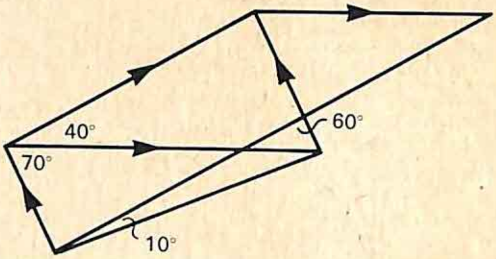
f Two dimensions

Copy the diagram and find all the angles in terms of p, q, r.



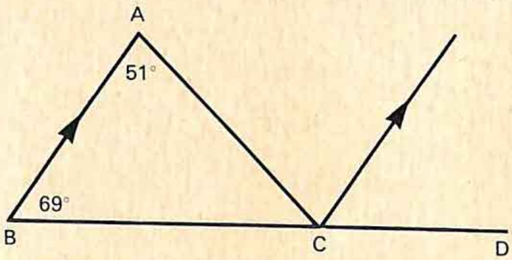
Exercise 8 In this exercise you must work out the results from what you are told. Measuring will not help you for the drawings are not accurate.

a Work out the angles of this figure.

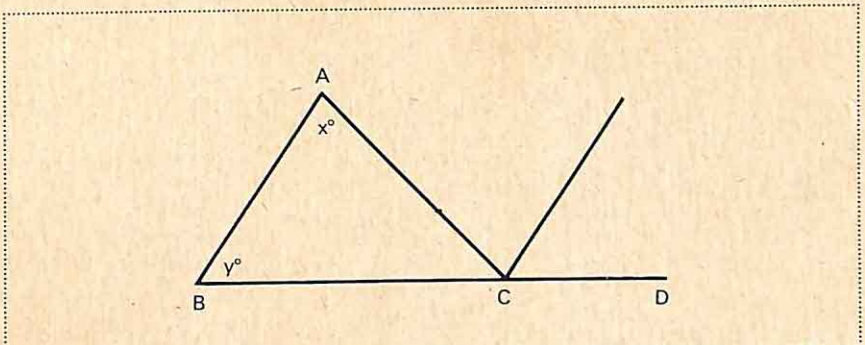


b Copy this diagram.

i Mark another angle which is 51° and another which is 69°.
ii Can you say anything about $\hat{A}+\hat{B}$ and $A\hat{C}D$?



c Copy this diagram.

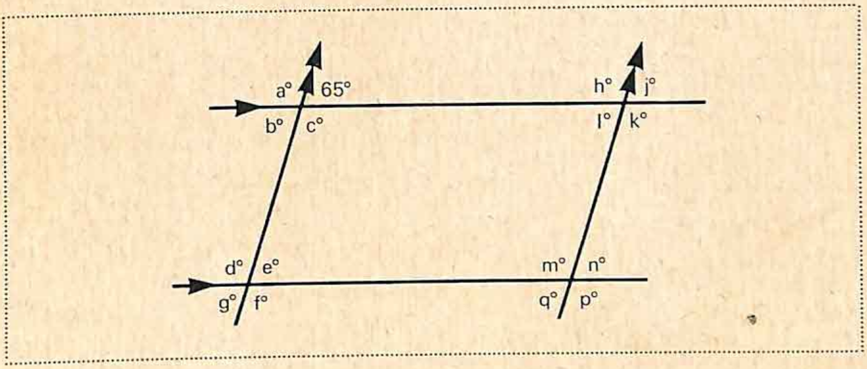


i Mark another angle which is $x°$ and another which is $y°$.

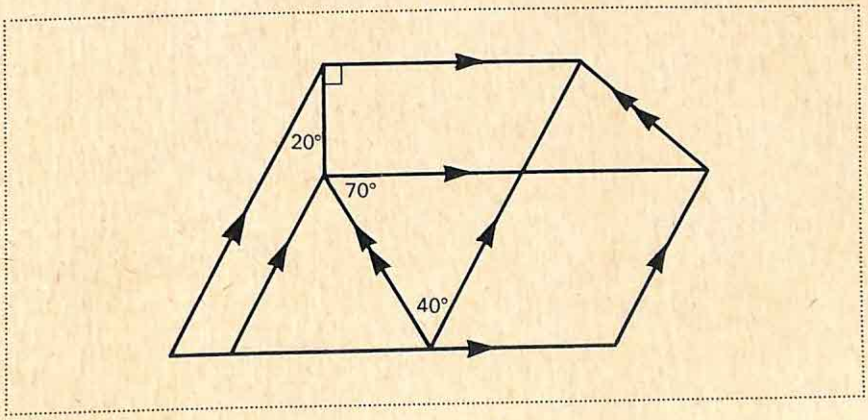
ii Can you say anything about angle A and angle B and angle ACD?

iii Is this true of *any* triangle?

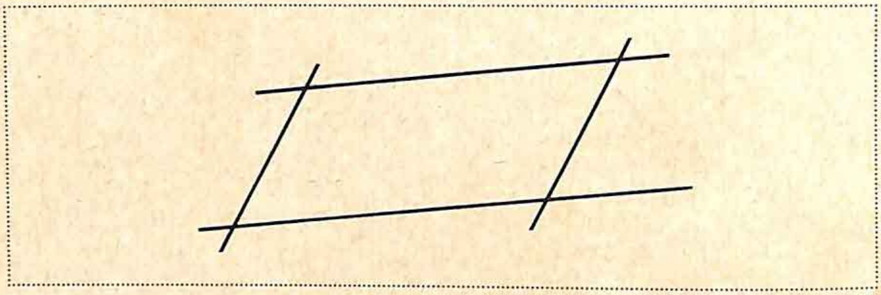
d Work out the angles in this figure.



e Work out the angles in this figure.



f This figure is called a *parallelogram.*



i If you were trying to tell someone about this figure, how would you describe it?

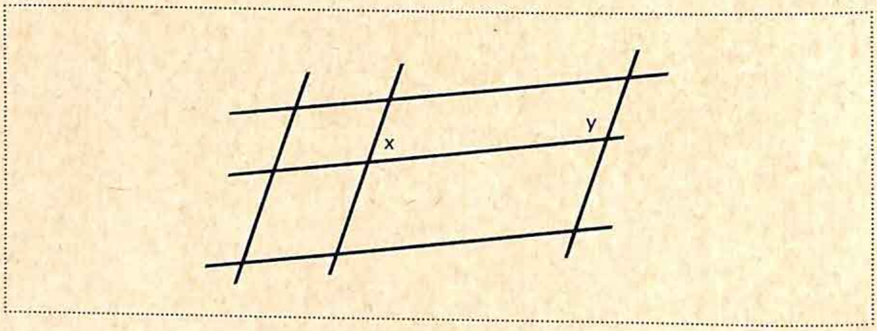
ii Look back at (d) and (e). What could you say about the angles at opposite vertices of this figure.

iii If you draw a diagonal line from one vertex to the opposite one, what can you say about the angles in the two triangles?

iv If you put in the other diagonal instead, what could you then say about the angles of the triangles?

v If you put in both diagonals, what could you say about the angles of all four triangles? (Be careful here – don't rush in!)

g Here is a figure made up of two sets of three parallel lines.



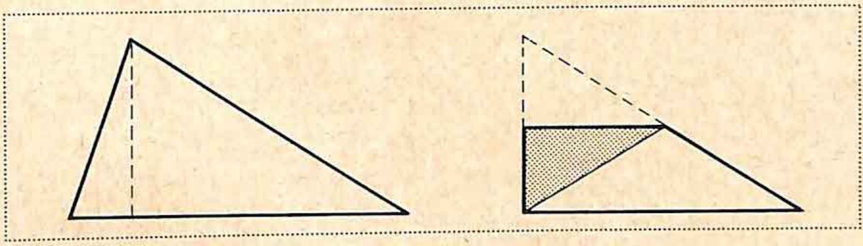
i How many parallelograms are there altogether in the figure?

ii Two angles are marked x and y. What can you say about $x+y$?

iii What can you say about the angles of all the parallelograms in the figure?

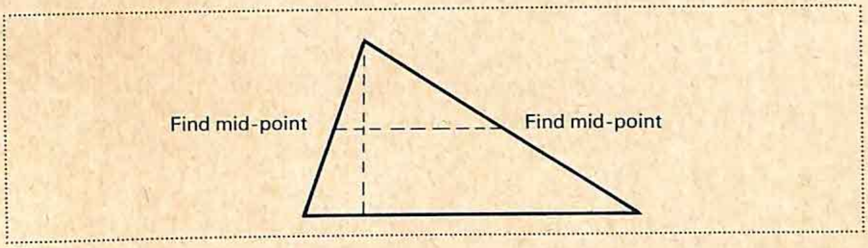
Exercise 9 For this exercise you will have to cut out of paper three triangles. Use plain paper and draw the triangle as we have shown on one piece. If you now put the three pieces on top of one another you can cut out three triangles all at once. You will have to be very careful indeed because if the paper slips you may not get straight sides on your triangles.

a Take your first triangle and fold a line perpendicular to the base.



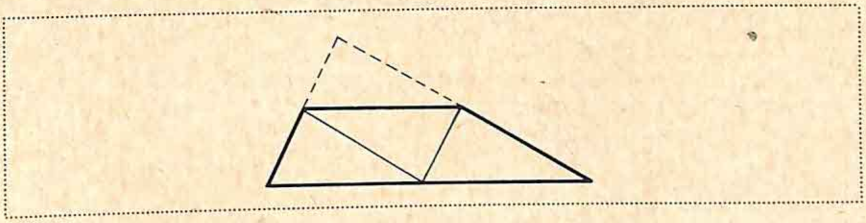
Keep the paper folded and bring down the vertex to the base.

Open out.



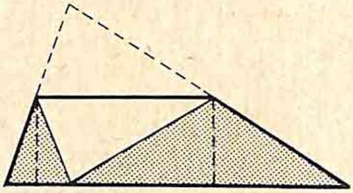
Find by measuring or folding the mid-points of the two sides indicated. What do you notice?

b Take your second triangle and find the mid-points of the sides. (You can do this most easily by folding.) Then fold across the mid-points until you have folds which look something like the diagram.



- i What do you notice about each of the three folds?
- ii How long are the folds? Compare them with the length of the sides of the triangle.
- iii How many parallelograms can you see in the figure?
- iv Cut off one of the small triangles and compare it with the other three. What do you notice?
- v Draw a sketch of your original triangle (not to scale), put in the lines which join the mid-points of the sides, and then mark the angles and line segments which you know are equal. Can you now see why there are parallelograms in the figure? Put arrows on the parallel lines.

c Take your third triangle and fold the line half way down the sides and parallel to the base. Keep it folded down.



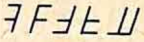
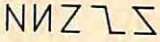
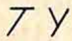
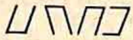
Now fold in the triangles along the dotted lines shown in the diagram.

i What shape do you make?
ii What can you say about the three angles of the triangle?
iii What can you say about two of the sides of each of the two triangles shaded in the diagram?
iv Are there any equal angles in those triangles?

Exercise 10 Do the whole of Exercise 9 again with a triangle of your own choosing. (Do not make it too small.)

Do you think you may have discovered some things about *all* triangles? Make a list of the points you now know. (The easiest way of putting them down on paper is to draw small diagrams.)

Summary

Shape	Name	Property
	Corresponding	Equal
	Alternate	Equal
	Vertically Opposite	Equal
	Adjacent	Supplementary (Add up to 180°)
	Allied	

Section 13 **More practice with frames**

Exercise 1 **a** In all of these questions, natural numbers only are to be used.

i $\square + 3 = 7$
ii $\triangle + 2 = 13$
iii $\nabla + 5 = 12$
iv $\square - 5 = 9$
v $7 + \square = 14$
vi $\nabla - 8 = 7$
vii $\triangle - 9 = 21$
viii $\triangle + \triangle = 16$
ix $\triangle + \triangle + \triangle = 18$
x $\square + \square = 24$
xi $\nabla + \nabla + \nabla + \nabla = 32$
xii $3 \times \triangle = 21$

b
i $5 \times \triangle = 25$
ii $\square \times 7 = 42$
iii $8 \times \square = 56$
iv $\square \times 6 = 54$
v $\square + 8 = 26$
vi $\triangle + 9 = 27$
vii $12 + \triangle = 53$
viii $14 \times \square = 42$
ix $\square - 13 = 13$
x $\square \div 2 = 13$
xi $13 \times 2 = \square$
xii $15 - \square = 8$

c
i $\square - 16 = 31$
ii $\triangle - 17 = 42$
iii $43 - \square = 36$
iv $43 - \triangle = 17$
v $\square - 5 = 5$
vi $\triangle \div 2 = 5$
vii $\nabla \times 2 = 10$
viii $18 - \square = 9$
ix $\triangle \div 2 = 9$
x $2 \times \nabla = 18$
xi $26 - \square = 13$
xii $26 \div \triangle = 13$

d
i $2 \times \nabla = 26$
ii $10 - \square = 8$
iii $12 - \triangle - \triangle = 6$
iv $12 \div \square = 6$
v $14 - 2 \times \nabla = 6$
vi $14 \div \triangle = 7$
vii $8 - \square - \square - \square = 2$
viii $16 - 3 \times \triangle = 4$
ix $18 - 3 \times \nabla = 0$
x $24 - 4 \times \nabla = 0$
xi $21 - \square - \square - \square = 0$
xii $20 - \triangle - \triangle = 0$

e
i $23 - \triangle - \triangle - \triangle = 2$
ii $36 - \square - \square - \square = 0$
iii $36 - \triangle - \triangle - \triangle - \triangle = 0$
iv $36 - \nabla - \nabla - \nabla = 6$
v $48 - 4 \times \square = 0$
vi $48 - 3 \times \triangle = 0$
vii $48 - 8 \times \nabla = 0$
viii $48 - 4 \times \square = 8$
ix $65 - 8 \times \nabla = 1$
x $52 - \triangle \times \nabla = 3$
xi $59 - \square \times \nabla = 2$
xii $\square - 6 \times 12 = 4$

Exercise 2 **a**

i $2+\square = 3\times 2$
ii $7+\triangle = 6+7$
iii $12+\triangledown = 15+12$
iv $\square+17 = 14+17$
v $4+\square = \triangle+5$
vi $7+\triangle = \square+6$
vii $10+\triangledown = \triangle+2$
viii $8+\square = \triangle+\triangledown$
ix $\square+27 = \triangledown+\square$
x $\square+\triangle = \triangle+\square$

b

i $2\times\square = 3+2$
ii $7\times\triangle = 6\times 7$
iii $\square\times 17 = 14\times 17$
iv $4\times\square = \triangle\times 5$
v $7\times\triangle = \square\times 6$
vi $8\times\square = \triangle\times\triangledown$
vii $\square+\triangle = \triangle+\square$
viii $3\times\square\times 5 = \square\times 3\times 5$
ix $\triangledown\times 3\times 9 = 9\times\triangledown\times 3$
x $5\times\triangle\times\triangledown = \triangledown\times\triangle\times 5$
xi $\triangle\times\square\times\triangledown = \triangledown\times\triangle\times\square$

Exercise 3 **a**

i $5+\square = 5$
ii $7+\triangledown = 7$
iii $6\times\square = \square$
iv $\triangle\times 8 = \triangle$
v $\square+\square = 0$
vi $12\times\square = 0$
vii $\triangle\times 0 = \triangledown$
viii $\triangle\times\triangledown = 0$
ix $\square+\triangledown = 0$

b

i $2\times(3+\square) = 14$
ii $5\times(\triangle+7) = 45$
iii $10\times(\triangledown+9) = 100$
iv $8\times(5+\triangledown) = 96$
v $8\times(4+\triangle) = 72$
vi $9\times(3+\triangle) = 72$
vii $13\times(\square+4) = 78$
viii $(\triangledown+2)\times 13 = 78$
ix $(\square+4)\times 6 = 78$
x $2\times(3+\square) = 2\times 3+2\times 2$

c

i $5(\triangle+7) = 5\times 6+5\times 7$
ii $(9+\triangledown)\times 10 = 9\times 10+10\times 8$
iii $(2+\square)\times 15 = 2\times 15+15\times 6$
iv $(2+7)\times 17 = 17\times 2+\square\times 17$
v $\square\times(2+5) = 5\times 2+5\times 5$
vi $5\times(\triangle+2) = 5\times\triangle+5\times 2$
vii $(3+\square)\times 8 = 3\times 8+\square\times 8$
viii $(\triangle+\square)\times 8 = \triangle\times 8+\square\times 8$
ix $(\triangledown+\triangle)\times 17$
$= \triangledown\times 17+\triangle\times 17$
x $(\square+\triangledown)\times\triangle$
$= \square\times\triangle+\triangledown\times\triangle$

Section 14 **The secret of equation-solving**

Exercise 1 Here is a game for you to play.

a i Think of a number.
ii Add 4.
iii Double your answer.
iv Subtract 3.
v Take away the number you first thought of.
vi Add 7.
vii Again subtract the number you first thought of.
The answer is 12.

b i Think of a number greater than 1.
ii Multiply it by 3.
iii Subtract 4.
iv Double the answer.
v Divide by 3.
vi Add on $2\frac{2}{3}$.
vii Subtract twice the number you first thought of.
The answer is 0.

How did I know what your answer was going to be? Let us put down your steps in each of the questions. We cannot write down your number because we do not know what it is. We will put a frame instead. These are your steps:

i $\square$
ii $\square+4$
iii $2(\square+4) = 2\square+8$
iv $2(\square+4)-3 = 2\square+8-3$
v $2(\square+4)-3-\square = 2\square+5-\square$
vi $2(\square+4)-3-\square+7 = 2\square+5-\square+7$
vii $2(\square+4)-3-\square+7-\square = 2\square-\square+12-\square = 12$

We can write some of these steps in a simpler form; this is done at the right-hand side.

Instead of a frame we will use an x this time.

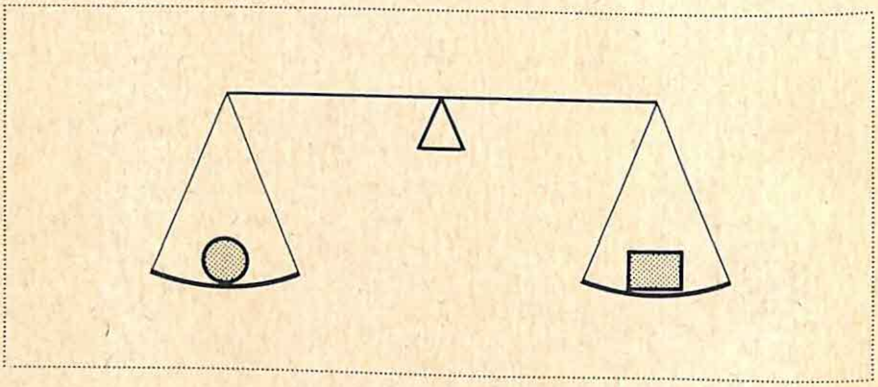
i x
ii $3x-4$
iii $2(3x-4) = 6x-8$

iv $$\frac{2(3x-4)}{3} = \frac{6x-8}{3} = \frac{6x}{3}-\frac{8}{3} = 2x-\frac{8}{3}$$

v $$\frac{2(3x-4)}{3}-2x = 2x-\frac{8}{3}-2x$$

vi $$\frac{2(3x-4)}{3}-2x+\frac{8}{3} = \frac{8}{3}-\frac{8}{3} = 0$$

Now you may not have understood all that, but by the time you have completed this section and Section 16 you will quite easily.
Let us begin by building up from something we all know about.
Imagine a pair of scales just balancing with an unknown weight on the left and a known weight on the other. Suppose we add 3 gm. to each pan. What happens? Is the balance upset? Add 5 gm. to each; does this upset the balance? Does it ever make any difference if we add the same amount to each pan?



Suppose we can take off a 3 gm. piece of the 'unknown' weight from the left-hand pan. If we also remove 3 gm. from the right-hand pan, will the balance be restored? Can we always restore the balance after removing part of the 'unknown' weight by removing an equal part of the known? We will try writing this down in symbols. For the 'unknown' weight we will write x; for the verb 'is balanced by', the sign $=$. Let us suppose the object is balanced by 8 gm. for which we will simply write 8. Then the statement:

'The unknown weight is balanced by 8 gm.' is written in symbols as

$x = 8$.

Add 3 gm. to each pan and the symbols read:

$x+3 = 8+3$

or

$x+3 = 11$.

Or take 5 gm. from each side and the sentence is:

$x - 5 = 8 - 5$

or

$x - 5 = 3.$

There are three different statements here: $x + 3 = 11$; $x - 5 = 3$; $x = 8$. The x in each case is the same thing and these equations express the same ideas; they are said to be *equivalent*. We can use = to say that two terms are equal, but we must never use = to say that two sentences or two equations are equivalent. The symbol for equivalence is ⇔ and this must only be used between equivalent *sentences*. For example: $x + 3 = 11 \Leftrightarrow x = 8$.

With the balance we can add or subtract quantities from both sides and it will still balance. The same rule applies to equations: we can add or subtract the same amount from both sides and the new statement(s) we obtain are all equivalent to each other.

If we look at the equation $x = 8$ or at the balance, we can see that $x + x = 8 + 8 \Leftrightarrow 2x = 16$. In the same way, $3x = 24$.

By the same argument $\frac{1}{2}x = \frac{1}{2}.8 \Leftrightarrow \dfrac{x}{2} = 4$. So we know that we can multiply or divide both sides as well.

Thus we see that with equations as with balances: we can add, subtract, multiply or divide both sides by the same amount and the results are all equivalent. This will help us to work with equations. For instance, if we know that $y + 4 = 10$, we also know that $y + 4 - 4 = 10 - 4 \Leftrightarrow y = 6$. Similarly if $z - 8 = 12$, then $z - 8 + 8 = 12 + 8 \Leftrightarrow z = 20$ and so on. (In the example remember that we can add and subtract in any order, $z - 8 + 8$ is the same thing as $z + 8 - 8$.) Also if $5a = 20$, then

$$5a \div 5 = 20 \div 5 \Leftrightarrow a = 4.$$

Exercise 2 Apply the method you have just learnt to these equations to solve them. You will not need the answers to the early ones for you will be able to see what they should be. You may check the answers to alternate questions from (b) onwards at the end of the section.

a
- i $x + 7 = 12$
- ii $x - 5 = 14$
- iii $2x = 12$
- iv $3x = 24$
- v $y \div 7 = 3$
- vi $\frac{1}{2}z = 14$

b
- i $2x + 4 = 14$
- ii $3x - 6 = 18$
- iii $3y + 12 = 36$
- iv $5z - 17 = 8$
- v $8c + 5 = 61$
- vi $5d - 13 = 47$
- vii $3k - 15 = 66$
- viii $9m + 11 = 110$

c i $3+x = 5$ iv $12-z = 11$ vii $20-3b = 5$
ii $4+x = 12$ v $12-z = 3$ viii $36-5c = 11$
iii $5-y = 2$ vi $12-2a = 6$

d i $5m+4 = 19$ iv $13-q = 8$ vii $3t-25 = 5$
ii $\frac{1}{2}n-4 = 6$ v $25-3r = 4$ viii $15u-75 = 30$
iii $2p-4 = 6$ vi $40-12s = 4$

Notice that all the equations you have been solving up to the moment have resulted in a natural number for the solution. You have been finding the set

$\{x \mid \text{some condition in } x;\ x \in \mathrm{N}\}$.

Is this always the case?

Exercise 3 Are the following statements true?

a $\{x \mid 3x = 5;\ x \in \mathrm{N}\} = \{\ \}$

b $\{x \mid 5-x = 7;\ x \in \mathrm{N}\} = \{\ \}$

c $\{x \mid \frac{1}{2}x = \frac{1}{4};\ x \in \mathrm{N}\} = \{\ \}$

If x in these examples were an element of some other set, it might well be the case that we could solve the equations. To solve the equations in (a) and (c), we need x to be an element of a set of numbers which we call the *rational numbers*. For the other, we need a different set which we shall meet in Book Two.

Let us look at the equation $3x = 5$. If we obey the rules we met at the beginning of this chapter, we obtain $x = 5 \div 3$ for the solution. In Section 16, we shall examine what this means.

Answers to Exercise 2

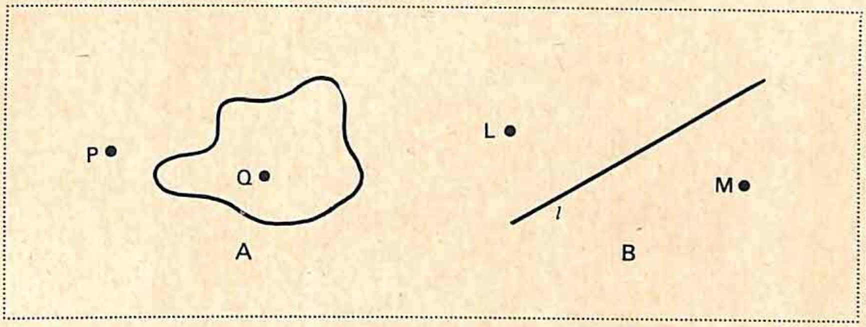
b i $x = 5$ v $c = 7$
iii $y = 8$ vii $k = 27$

c i $x = 2$ v $z = 9$
iii $y = 3$ vii $b = 5$

d i $m = 3$ v $r = 7$
iii $p = 5$ vii $t = 10$

Section 15 **More drawing**

Exercise 1 Look at the two drawings A and B. They are both drawn on infinite planes. They are both *plane figures.*



a Into how many sets does the line in A partition the plane?

b We will call a set in the plane a *region* or a *line*. When it is not a line, it is a region. How many regions are there in Diagram A?

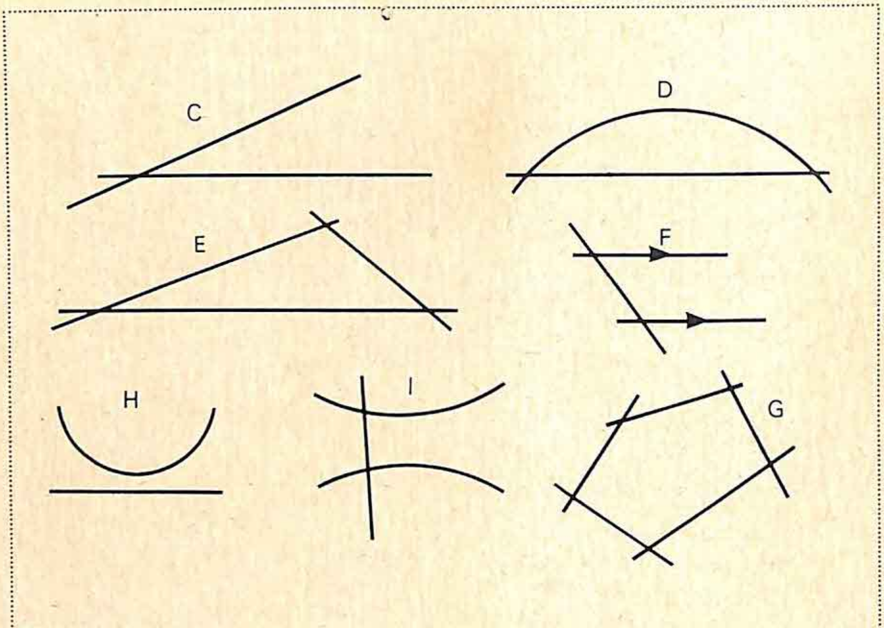
c There is a point P marked in one of the regions of Diagram A. Suppose it can move about anywhere in that region. Is there at least one direction in which it can go without coming to the edge of the region? (Make a sketch to illustrate your answer.)

d There is a point Q marked in another region of Diagram A. Is there at least one direction in which this point can move without meeting a boundary of the region?

e Look at Diagram B.
  i How many sets are there?
  ii How many regions are there?
  iii Can each of L and M move in at least one direction without meeting a boundary?

f There is only one line in each of Diagrams A and B. They have very different properties. What is the difference between the lines which makes these properties so different?

g If a region is limited in size in all directions, it is called *finite*; otherwise, it is *infinite*. Look at Diagram C. How many regions are (i) finite and (ii) infinite?



h You may find the Diagrams C–I help you here.

- i Can one line enclose a finite region?
- ii When can it not do so?
- iii Can two lines fully enclose a finite region?
- iv When do they not do so?
- v What is the smallest number of straight lines which will fully enclose a finite region?
- vi When does this fail?
- vii Can you enclose a finite region with more than this number of lines?
- viii Can you draw a figure consisting of five straight lines which do not enclose a finite region?

Exercise 2 **a** Look at sketch E again. We usually call this region a *triangle*.

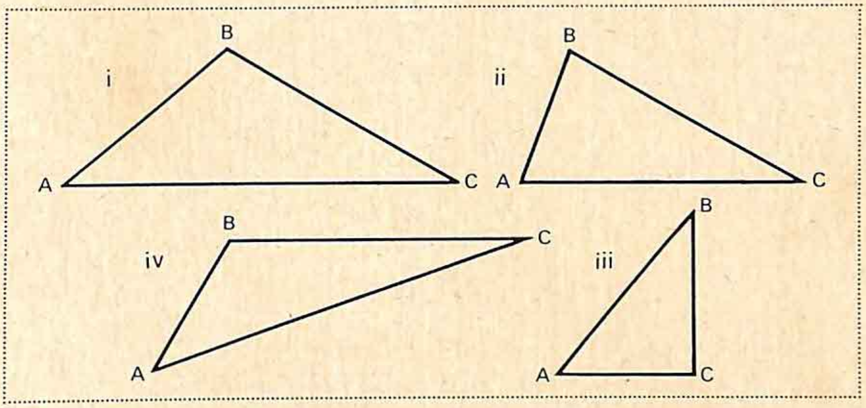
- i What other words can you find which begin with 'tri-'?
- ii What does the 'tri-' mean?
- iii Look up 'trigon' in a good dictionary.

b Find six examples of triangles in your room.

c Where do you see triangles outside the school? Find six examples.

d Look at your two set squares. In one important way they are alike. In what way?

e Look at these four drawings.

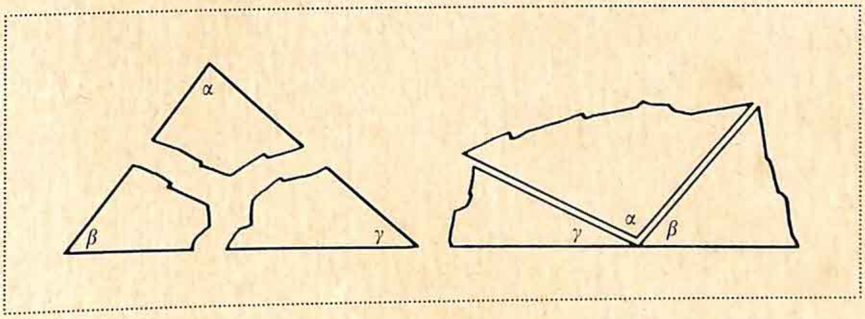


i In what way are I, II and III quite different?
ii Why is IV like I?

f Say whether you could draw triangles with the following kinds of angles and explain your answers:

i three acute angles
ii one obtuse angle and two acute angles
iii two obtuse angles, one acute angle
iv three obtuse angles
v one right angle, two acute angles
vi two right angles, one obtuse angle

Exercise 3 Cut out a triangle in adhesive paper. Mark the angles α, β, γ. Tear it as shown in the diagram.



a Rearrange the torn pieces as we have done, and stick them down.

b How many ways can you arrange the torn pieces on a straight line? Cut out some more triangles to show this.

c What does this tell you about the angles of a triangle?

Exercise 4 **a** Draw five triangles. Try to make them as different as possible. Use a ruler to draw the sides and make these at least 4 cm. long. Number your triangles I, II, III, IV, V. Name the angles A, B, C. Measure the angles as accurately as you can and make a list of the results.

Triangle	$\hat{A}$	$\hat{B}$	$\hat{C}$	$\hat{A}+\hat{B}+\hat{C}$
I				
II				
III				
IV				
V				

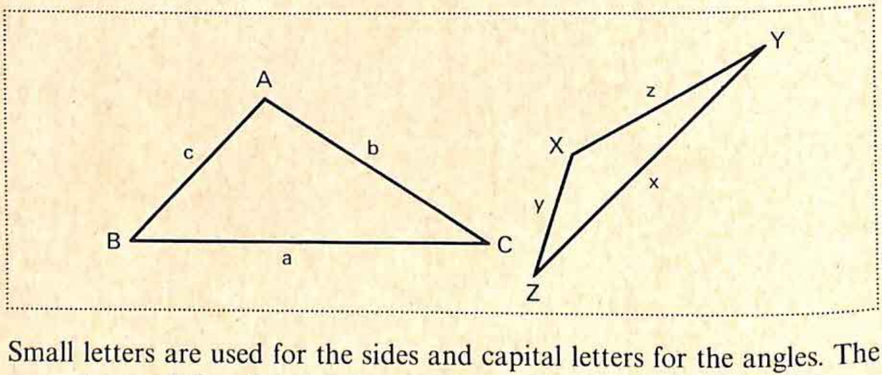
Fill in the last column by adding the results of the others.

b i Do you think your measurements are absolutely exact?
ii What does this exercise tell you about the angles of a triangle?

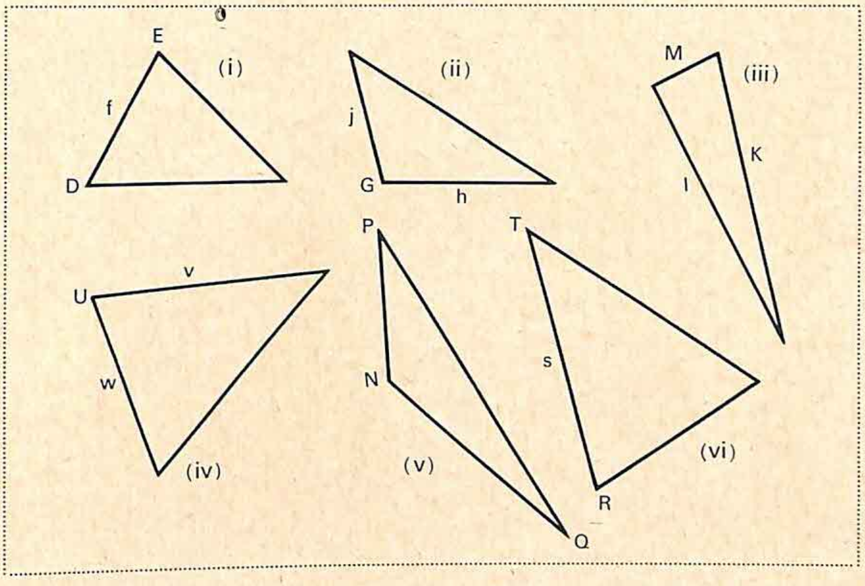
Exercise 5 For this exercise you can use the triangles you drew for the last exercise. Measure the sides of the triangles.

a i What can you say about the largest side and the largest angle?
ii What can you say about the smallest angle and the smallest side?
iii What can you say about the middle-sized angle and the middle-sized side?

b It is very convenient to name the sides of a triangle in this way:



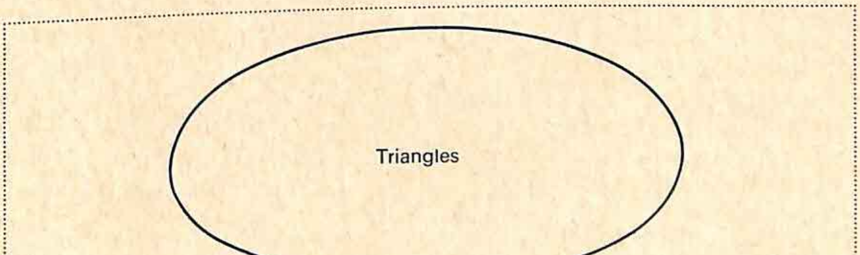
Small letters are used for the sides and capital letters for the angles. The same letter of the alphabet is used for a side and an angle which are opposite to one another. Sketch the following triangles and name the sides and angles not already named.



c Is it possible to draw triangles with the following sides? Explain your answers.

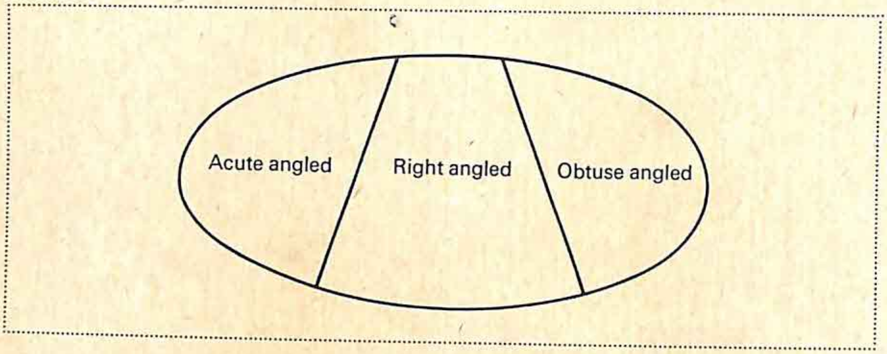
i 20 cms., 30 cm., 40 cms.
ii 1 ins., 2 in., 4 in.
iii 12 cm., 13 cm., 20 cm.
iv 8 cm., 12 cm., 8 cm.
v 5 in., 12 in., 5 in.
vi 570 cm, 570 cm., 2 cm.
vii $A = 20°$, $B = 70°$, $C = 90°$, $a = 30$ cm., $b = 50$ cm., $c = 40$ cm.

Let us think now about the *set of all triangles.*

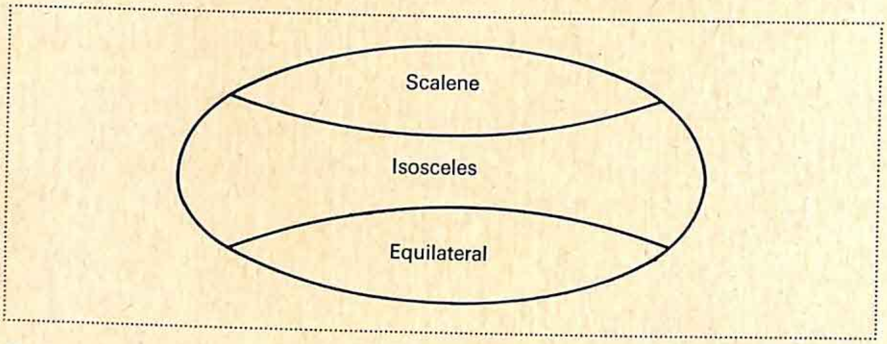


We can partition the set into three parts:

acute angled triangles (all the angles are acute),
obtuse angled triangles (one angle only is obtuse),
right angled triangles (one angle only is a right angle).

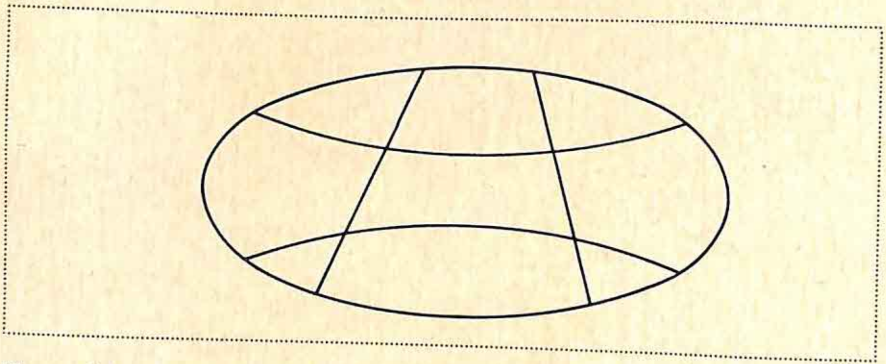


This partition is based on the angles in the triangles. We can make another partition by thinking about sides.



There are again three disjoint subsets:

triangles with all their sides different lengths (scalene);
triangles with only two equal sides (isosceles);
triangles with three equal sides (equilateral).



If we pick up the third diagram and place it on the second we find a diagram divided into nine parts. It may be that some of these subsets are empty because the two qualities which they are supposed to have are contradictory.

Exercise 6 You will need to think very carefully and to do a lot of rough sketches to be sure of the answers to these questions. Find out for each what the triangle looks like; if the set is empty, say so.

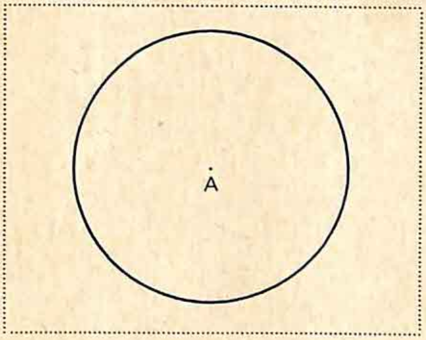
a i scalene, acute angled triangle
ii isosceles right angled triangle
iii equilateral obtuse angled triangle
iv equilateral acute angled triangle
v obtuse angled scalene triangle
vi equilateral right angled triangle
vii obtuse angled isosceles triangle

b We have forgotten two of the possibilities in (a). What are they? Make the sketches as before.

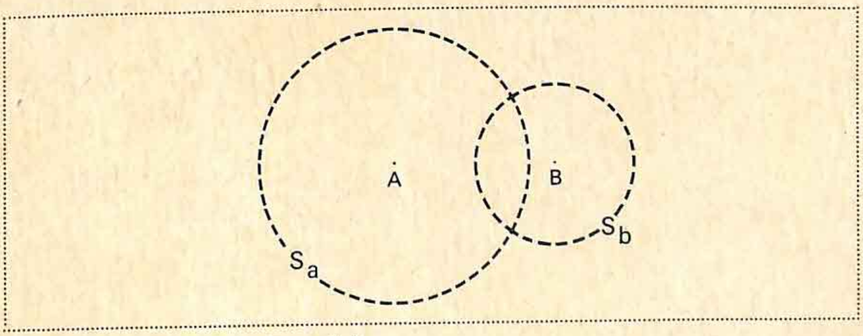
c Can you now draw a partition of the set of triangles which neither leaves anything out nor includes any empty sets?

Exercise 7 **a** Look at this drawing: it illustrates a set of points and another point marked A. This is a plane figure.

i What can you say about the set of points and the point A?
ii If you think of the set of positions of a moving point, what name do you give to the set?
iii What is the special name of this path?

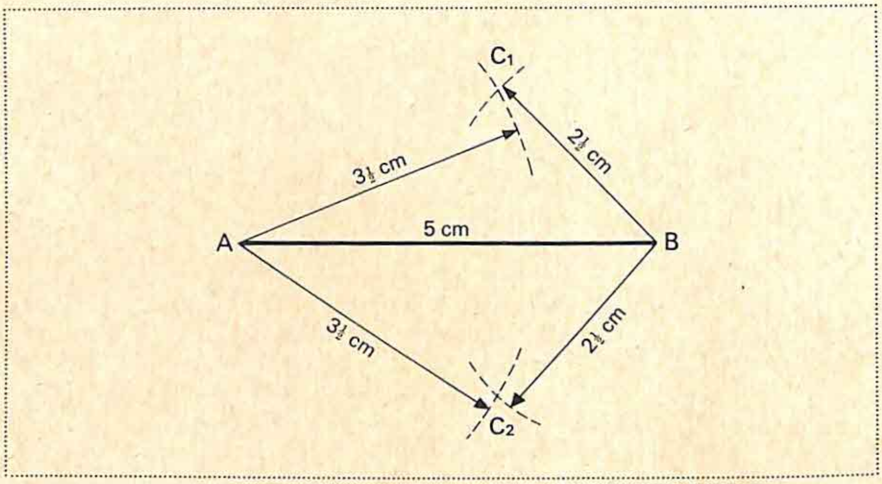


b Look now at this drawing. It shows the curve S_a round the point A as before, but it now shows another curve S_b round the point B as well.



i What sort of set is $S_a \cap S_b$?
ii what is $n(S_a \cap S_b)$?
iii If $C \in S_a \cap S_b$, what can you say about C and the two points A and B?

c If we rub out part of the drawing we can use what is left to draw a triangle when we know the three sides.



Remember that the small curves or arcs are drawn with a pair of compasses because they are really parts of circles with their centres on A and B.

i How many triangles are possible with three sides given?
ii How are these triangles related?

d Draw triangles with the following sides, in each case obtaining two different results on the same diagram. Say which sort of triangle you have in each example (using the categories of Exercise 6).

i 3 in., 2 in., 4 in.
ii 7 cm., 8 cm., 10 cm.
iii 15 cm., 4 cm., 17 cm.
iv 8 cm., 8 cm., 6 cm.
v 6 cm., 8 cm., 10 cm.
vi 5 cm., 5 cm., 5 cm.
vii 5 cm., 5 cm., 7 cm.
viii 7 cm., 7 cm., 14 cm.

In each example you have two triangles: they are 'the same' yet 'different'. If you cut one out and turn it over, it will fit exactly over the other. If you place a mirror along the common side, you will see a relationship between the two. One is an *image* of the other under *reflection.*

There is a special word for this 'same only different' in geometry. If two figures can be put on top of one another and made to fit exactly, even if you turn one over to get the fit, they are said to be *congruent*. We are at present interested in congruent triangles only.

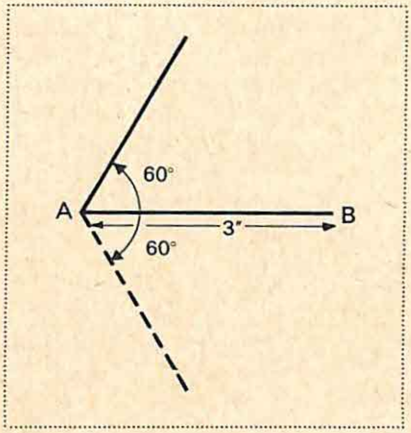
Now here is a problem: suppose I give a whole class the same pieces of information about a triangle, what is the smallest amount of information I must give to make sure that all the triangles they draw are congruent? This is the important point we will settle now.

Exercise 8 **a** If we give three sides does everyone draw triangles congruent to each other? (Before you answer look back to Exercise 5 c ii.)

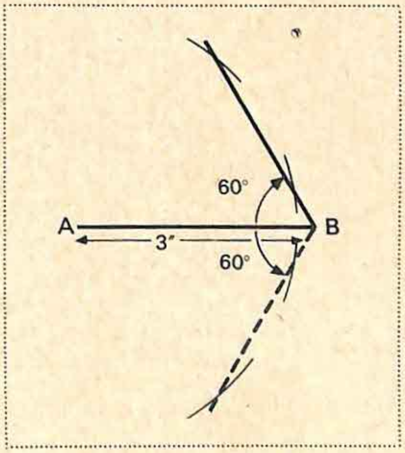
b When do we find that we cannot draw a triangle at all?

c Draw AB = 3 in. and angle A = 60°. Mark the position of C 2 in. from A. How many triangles can you get? Are they congruent? Is one a reflection of the other?

This given angle lies between the two given sides; it is said to be an *included* angle.



d Now try a non-included angle. Draw AB = 3 in. and at B an angle of 60°. With centre A and radius $2\frac{3}{4}$ in. draw the parts of the circle which will identify C. How many triangles are there? Are they *all* congruent? Are any of them reflections of any other? What would happen if we tried to make angle C the 60° angle? How could we draw the triangles?



e Now try to draw a triangle with AB = 3 in., AC = $2\frac{3}{4}$ in. and angle C = 60°. You will have to experiment to find out how to draw it.

f Now we will try drawing triangles in which we are given a side and two angles, say 2 in., 45° and 60°. Draw sketches to show three different triangles with these measurements. Does it matter which side is 2 in.?

If two people wish to draw congruent triangles given this information, what additional fact must they be given? Draw accurately the three different triangles.

g What would happen if you were told to draw a triangle given three angles only? How many triangles could you draw? Would they be the same shape? Would they be the same size? Would they be congruent?

We need a shorthand to say what sort of set of facts we are given. When we know that this information is given, we know that the triangles we have are congruent. This is an important piece of information for the sort of geometry we shall often be doing in our future work. Here is a summary of what we know so far.

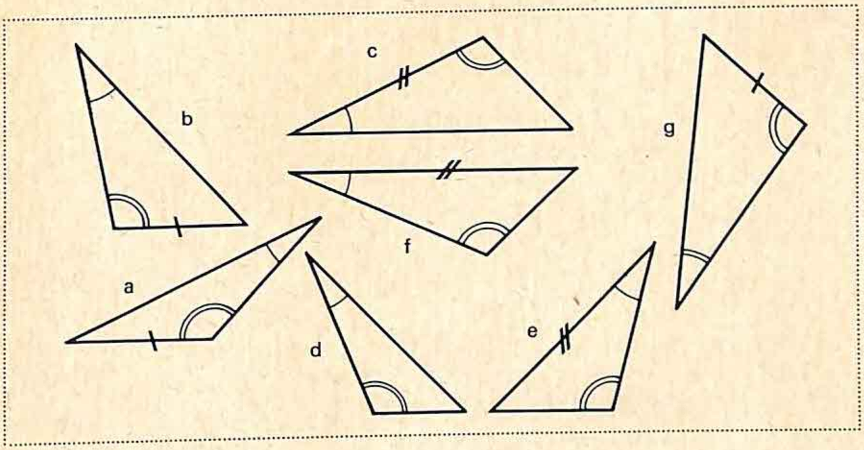
	Facts given	Shorthand for facts given	No. of possible triangles	Useful to identify congruence?
1	3 sides (remember the limitation)	SSS	1 and its reflection	Yes
2	2 sides and 1 included angle	SAS	1 and its reflection	Yes
3	2 sides and 1 non-included angle in a given position	ASS	2 different triangles and reflections	No
4	1 side and 2 angles (provided the position of side is given)	AA corr S	1 and its reflection	Yes
5	3 angles	AAA	any number	No

In line 3 of the table, we have another possibility. Try drawing a triangle in which the non-included angle is 90°. How many triangles do you get? What happened to the other triangles you expected to get? What happens if you use an angle less than or greater than 90°?
From this we see that we have to add an extra line to our table:

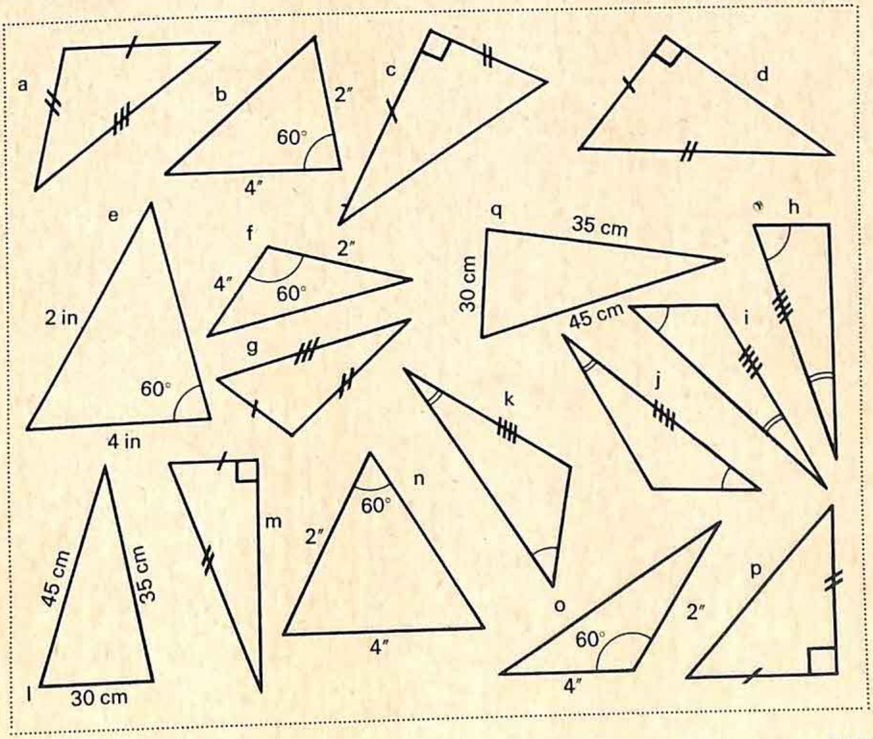
3A	2 sides and a non-included right angle	RHS	1 and its reflection	Yes

RHS stands for *right-angle, hypotenuse and side*; the longest side of a right-angled triangle is called a *hypotenuse.*

Exercise 9 Say which of these triangles are congruent, AA corr S.

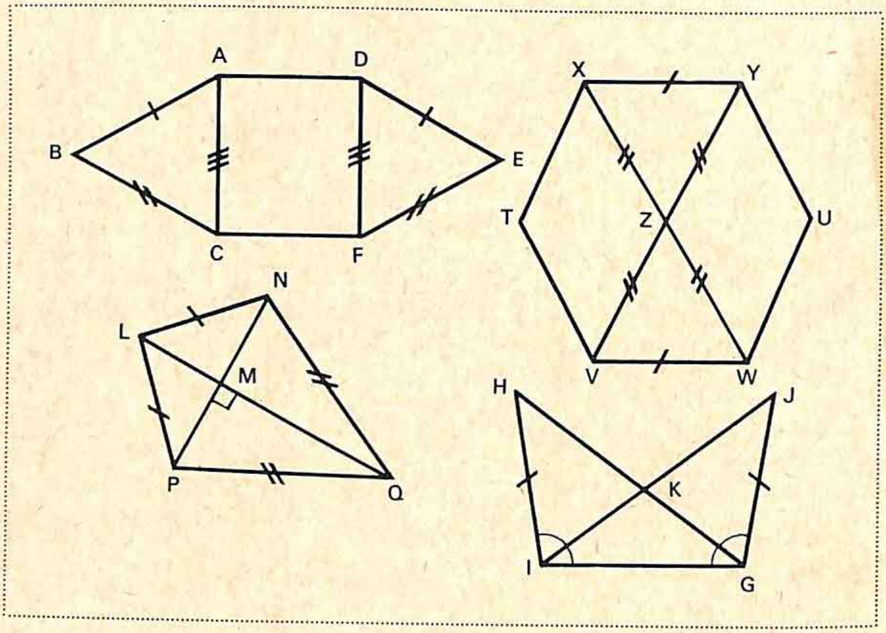


Exercise 10 Here is a set of triangles and the given information is marked on each one. They are sketches so they are not drawn either to scale or with accurate angles. Look at the information on them and not what they look like. Partition the set into sets of congruent triangles, and beside each set use the shorthand to say what sort of congruence it is.



The symbol ≡ means 'is congruent to'. '△ABC ≡ △CDE' is read 'triangle ABC is congruent to triangle CDE'.

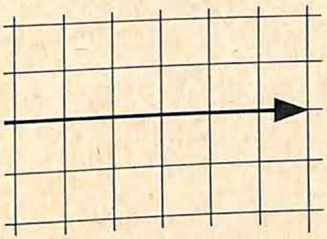
Exercise 11 Here are four figures lettered a, b, c, d. Write true statements for each about the congruent triangles in the figure. For one of the diagrams you can write two statements. After each, in parentheses, write the kind of congruence.



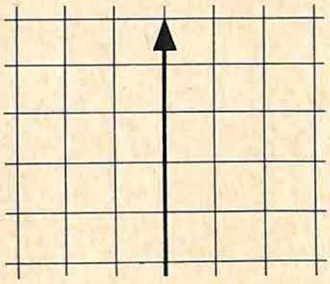
Section 16 **Some new numbers**

Let us invent some new numbers. Each one will be in a pair of brackets or parenthesis, (,). Inside the brackets there will be two natural numbers, for instance (3, 2) or (6, 8) or (27, 76). Such an arrangement of numbers is called either an *ordered numbers pair* or a *couple.*

We are going to begin by making a graph of these couples. We shall use squared paper and measure the first number sideways to the right:



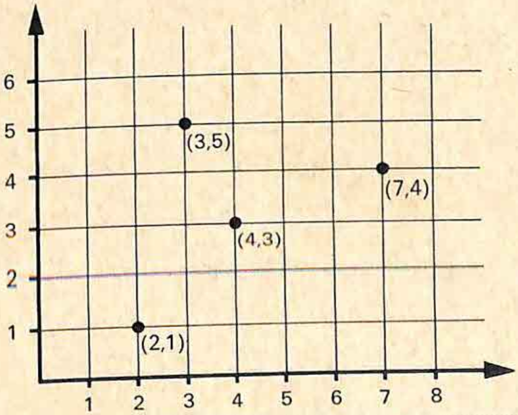
The first number is measured this way



The second is measured this way

and the second upwards:

We shall use equal distances on each axis to represent the number 1. The numbers will then be represented as points. Here are the numbers (2, 1), (3, 5), (4, 3), (7, 4) plotted for you.



Exercise 1 **a** On a piece of squared paper draw two number lines as we have done and number them to at least 10. Now plot these points:

i (4, 8)	iv (6, 10)	vii (2, 4)
ii (3, 5)	v (1, 2)	viii (9, 3)
iii (3, 1)	vi (6, 2)	

b i Draw a line from where the number (0, 0) would be (the *origin*) to the point (9, 3). What do you notice?

ii Draw a line from the origin to the point (6, 10). What do you notice?

iii Draw a line from the origin to the point (4, 8). What do you notice?

iv You have now partitioned the set of points you plotted into three sets. What have the elements of each set in common?

c When two of our new numbers lie on the same line from the origin, we shall say they are *equal*. Thus (6, 10) = (3, 5).
Write down the equalities you have just discovered.

d On another pair of number lines plot the point (3, 2). Draw a line from the origin to pass through this point.

i All the couples on this line are equal. Write the word EQUAL along the line.

ii In the region above the line write LESS THAN. Couples in this region will be said to be less than couples on the line.

iii In the region below the line write GREATER THAN. Couples in this region will be said to be greater than couples on the line.

These are, you will be saying, very peculiar numbers. So they are but very shortly you will recognize them as something you have met many times before.

Exercise 2 **a** How many times greater than 3 is 6?

b How many times is 4 smaller than 12?

c What do I have to multiply 7 by to get 21?

d What is the result of dividing 24 by 3?

e Each of these very simple questions could be rewritten in the same kind of way, as an equation. (c) for instance is $7x = 21$. Rewrite the others as equations.

If we ask how many times greater is 8 than 4, we are using a special relationship between the numbers 4 and 8. It can be expressed not only as an equation but as a division, or it can be expressed as a *ratio* of the two numbers. The symbol used for ratio is the colon : . So we write 8 : 4 which means 'the ratio of 8 to 4' and is read out as '8 is to 4'.

What can you say about 10 : 5 and 6 : 3? They are equivalent. 10 is the same number of times greater than 5 as 6 is than 3. They are both equivalent to 2 : 1. In fact we can write 10 : 5 = 6 : 3 = 2 : 1.

Exercise 3 From the following ratios, select pairs which are equivalent and write down the appropriate sentences (five altogether).

12 : 6; 4 : 2; 8 : 1; 15 : 5; 12 : 4; 24 : 3; 1 : 4; 2 : 12; 2 : 8; 3 : 18.

Another way of writing ratios is to place one number above the other, e.g. $\frac{12}{4}$ or $\frac{7}{14}$, and the kind of sentences which you write as $\frac{12}{4} = \frac{3}{1}$ or $\frac{7}{14} = \frac{21}{42}$. When ratios are written in this way they are called *common fractions* or, better, *rational numbers.* The sentences we have written above tell us about the equivalence of rational numbers, or the equivalence of fractions.

Exercise 4 **a** Rewrite the five sentences of Exercise 3 as the equivalence of fractions.

b Put the correct numbers into the 'boxes' in the following statements:

i $\frac{4}{8} = \frac{\square}{12}$ ii $\frac{2}{6} = \frac{6}{\square}$ iii $\frac{\square}{3} = \frac{5}{15}$ iv $\frac{2}{\square} = \frac{5}{25}$

c Find the solutions of the following:

i $\frac{1}{6} = \frac{10}{x}$ iii $\frac{8}{1} = \frac{z}{24}$ v $\frac{b}{1} = \frac{85}{5}$

ii $\frac{4}{16} = \frac{y}{32}$ iv $\frac{16}{1} = \frac{48}{a}$

d Find five more fractions equivalent to each of these, writing them as a set of equivalent fractions beginning with the one containing the smallest numbers.

i $\frac{3}{5}$ ii $\frac{5}{8}$ iii $\frac{8}{12}$ iv $\frac{7}{8}$

Many of the fractions we have written so far can be written as equivalent to a ratio or fraction containing 1. For instance: $\frac{2}{6} = \frac{1}{3}$ and $\frac{15}{3} = \frac{5}{1}$. Can this be done with all ratios? Can it be done with 7 : 4 or 9 : 13? Can you rewrite $\frac{3}{11}$ or $\frac{15}{7}$ as a fraction containing 1?

It is useful to have special names for these when they are fractions (but not when they are ratios written with a colon). Fractions like $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{392}$ are called *unit fractions.*

Fractions like $\frac{3}{1}$, $\frac{2}{1}$, $\frac{392}{1}$ are usually shortened to 3, 2, 392 by the omission of the 1 on the bottom. Fractions like $\frac{3}{2}$, $\frac{11}{5}$, $\frac{5}{91}$, where there is no whole number which will divide into each of the numbers in the fraction are said to be *reduced to their lowest terms.*

Exercise 5 Reduce each of the following to its lowest terms, using the shortened form where possible.

a $\frac{15}{51}$
b $\frac{8}{56}$
c $\frac{204}{4}$
d $\frac{81}{9}$
e $\frac{11}{2}$
f $161:7$
g $42:6$
h $7:84$
i $57:3$
j $4:72$
k $\frac{14}{36}$
l $\frac{25}{5}$
m $\frac{63}{81}$
n $\frac{12}{108}$
o $\frac{63}{27}$
p $\frac{24}{36}$
q $\frac{30}{25}$
r $\frac{114}{150}$
s $\frac{45}{100}$
t $\frac{135}{90}$

Some of the fractions were top heavy, the last for instance. Fractions can be like this and an answer you give which is top heavy is not wrong because of that.

Have you noticed any connexion between rational numbers and the new numbers we began with?

Here is another idea for you to think about. How did we obtain the shortened form for fractions like $\frac{12}{4}$? Clearly, $\frac{12}{4}$ can mean 12 *divided by* 4. Why is $\frac{3}{1}$ shortened to 3? $\frac{12}{4} = 3$ can be read out as '12 divided by 4 equals 3' or as 'the ratio of 12 to 4 is the same as the ratio of 3 to 1.' In the same way $\frac{1}{5}$ can mean the ratio of 1 to 5, or 1 divided by 5. Again, $\frac{2}{5}$ can mean a ratio of 2 to 5 or $2 \div 5$, and so on.

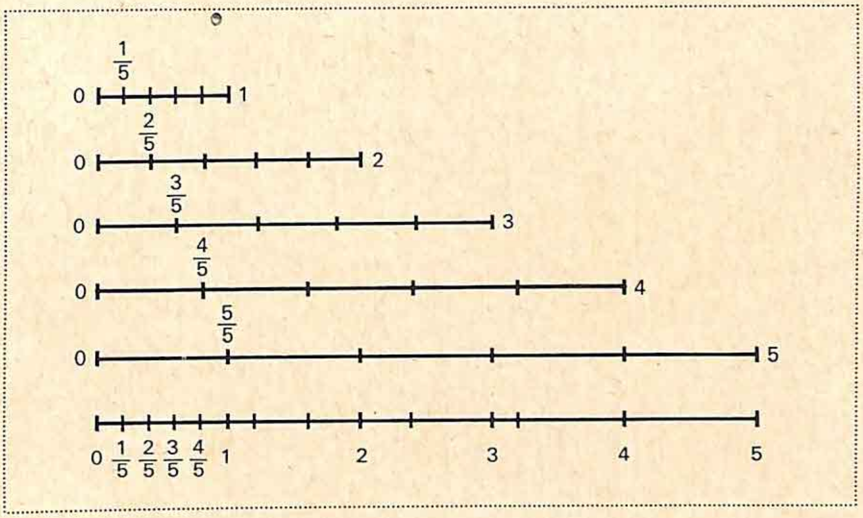
Exercise 6 On a sheet of $\frac{1}{10}$ in. graph paper draw a series of parallel lines an inch apart. The first line is to be 1 in. long, the second 2 in. and so on up to 6 in. Draw two lines 6 in. long. Divide each of your first six lines into five equal parts. Label the first division in each by the correct fraction. Now use your last line to display all of these subdivisions at once. (We have done this, with one less line, below.)

a Why are the divisions at equal intervals in the first inch in the last line but not afterwards?

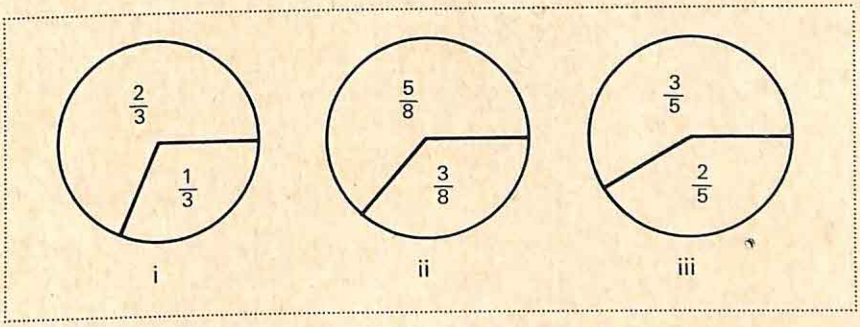
b How could you complete the picture with all the multiples of $\frac{1}{5}$?

c Name the remaining points on the last line.

d Do a similar drawing to show the build up of $\frac{1}{3}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{5}{3}$, 2, etc.



Another kind of diagram which is useful for some purposes is the 'pie' diagram. The circle represents 1; the sectors, fractions.



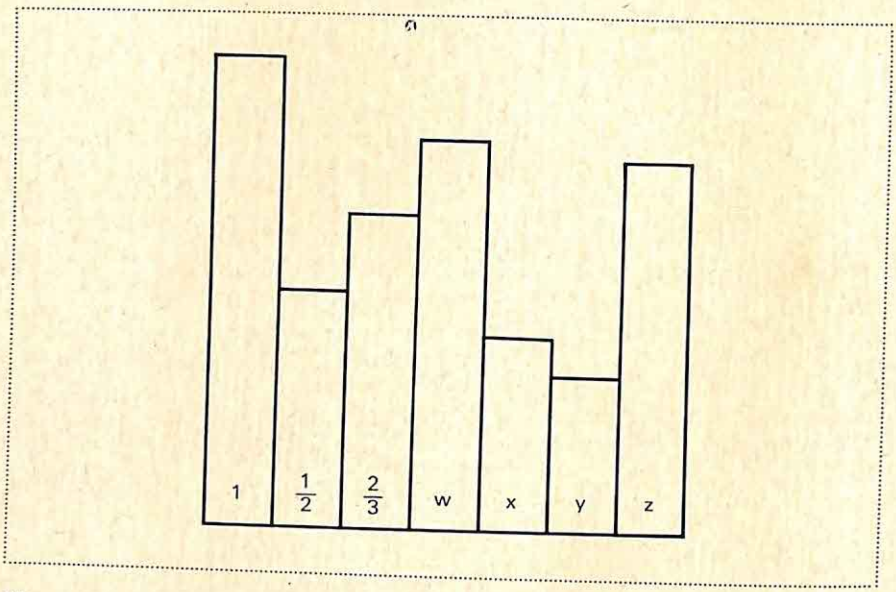
Exercise 7 **a** What are the angles between the straight lines in (i), (ii), and (iii) above? (Give reflex as well as obtuse angles, and find them by calculation.)

b What non-reflex angles would we have to draw to illustrate $\frac{1}{6}$, $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{36}$, $\frac{1}{20}$, $\frac{1}{18}$, $\frac{1}{8}$, $\frac{1}{2}$, $\frac{1}{120}$, $\frac{1}{72}$?

c What angles would we need for the following fractions? $\frac{5}{6}$, $\frac{4}{5}$, $\frac{29}{36}$, $\frac{11}{72}$?

Yet another kind of diagram we could make would be a block diagram: (see page 140.)

Exercise 8 **a** If the first three blocks represent the quantities marked on them, find by measurement what *w*, *x*, *y*, *z* stand for.



b How many times greater is $\frac{2}{3}$ than $\frac{1}{3}$?

c Make a block diagram to show 1, $\frac{3}{8}$, $\frac{1}{8}$, $\frac{5}{8}$, $\frac{3}{4}$.

d How many times greater than $\frac{1}{8}$ is $\frac{3}{8}$?

e We can express $\frac{3}{8}$ as $3 \times \frac{1}{8}$. How can we express $\frac{5}{8}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{5}{6}$?

Exercise 9 Look back to the diagram printed with Exercise 6.

a Check from the diagram that

i $2 \times \frac{2}{5} = \frac{4}{5}$ ii $4 \times \frac{3}{5} = \frac{12}{5}$

b Find

i $4 \times \frac{6}{5}$ ii $3 \times \frac{8}{5}$ iii $5 \times \frac{2}{5}$ iv $7 \times \frac{3}{5}$

c Calculate these. You may check alternate answers at the end of the section.

i $2 \times \frac{2}{3}$
ii $9 \times \frac{3}{4}$
iii $6 \times \frac{5}{6}$
iv $3 \times \frac{8}{7}$
v $4 \times \frac{4}{5}$
vi $7 \times \frac{9}{8}$
vii $\frac{4}{9} \times 5$
viii $\frac{3}{8} \times 2$
ix $\frac{5}{6} \times 8$
x $\frac{12}{13} \times 6$
xi $4 \times \frac{15}{17}$
xii $12 \times \frac{31}{29}$

d By looking at the diagram of Exercise 6, or by calculation give the values of

i $\frac{2}{3}$ of 5
ii $\frac{1}{2}$ of $\frac{2}{3}$
iii $\frac{2}{3}$ of $\frac{1}{2}$
iv $\frac{2}{3}$ of $\frac{4}{3}$
v $\frac{4}{3}$ of $\frac{2}{3}$
vi $\frac{7}{8}$ of 6

Alternate answers are at the end of the section.

e When we have a set of equivalent fractions, it is often convenient to use the one which contains the smallest numbers. This is said to be the *simplest form* or the *lowest terms.*

Put each of the following into its simplest form. (Alternate answers are at the end of the section.)

i $\frac{4}{6}$
ii $\frac{6}{8}$
iii $\frac{3}{9}$
iv $\frac{9}{15}$
v $\frac{18}{12}$
vi $\frac{12}{20}$
vii $\frac{7}{14}$
viii $\frac{10}{16}$
ix $\frac{15}{9}$

f If you look at the diagram of Exercise 6 again, you will see that $\frac{4}{3}$ is the same thing as $1+\frac{1}{3}$. We can agree to leave out the + sign here just as we do with 23 (twenty-three), and write $1\frac{1}{3}$. Sometimes people say that fractions like $\frac{4}{3}$ which are top heavy are *improper fractions* although there is nothing improper about them. Then expressions like $1\frac{1}{3}$ are called *mixed numbers* although there is not really anything mixed about them!

Put each of the following into an alternative form. (Alternate answers are at the end of the section.)

i $\frac{5}{4}$
ii $\frac{7}{2}$
iii $\frac{8}{3}$
iv $\frac{12}{5}$
v $1\frac{1}{2}$
vi $2\frac{3}{4}$
vii $\frac{5}{6}$
viii $\frac{14}{6}$
ix $4\frac{7}{8}$
x $5\frac{9}{10}$
xi $\frac{16}{7}$
xii $\frac{18}{8}$

We can now come back to the point we reached in the last section. We started with these new numbers in the hope that they would make it possible for us to solve certain equations.

Exercise 10

Solve the following equations. Answers to alternate ones are at the end of the section.

a i $5x = 8$
ii $9x = 7$
iii $4x = 13$
iv $17x = 21$
v $15x = 25$
vi $27x = 63$
vii $14y = 21$
viii $12z = 68$
ix $14a = 91$
x $26b = 104$
xi $27c = 126$
xii $36d = 90$

b Find numbers to put into these frames.

i $\frac{6}{5}\times\square = \frac{3}{5}$
ii $\frac{4}{7}\times\square = \frac{2}{7}$
iii $\frac{5}{9}\times\square = \frac{1}{9}$
iv $\frac{2}{3}\times\square = \frac{1}{9}$
v $\frac{3}{4}\times\square = \frac{2}{4}$
vi $\frac{3}{8}\times\square = \frac{2}{8}$
vii $\frac{5}{8}\times\square = \frac{3}{8}$
viii $\frac{4}{7}\times\square = \frac{3}{7}$
ix $\frac{12}{8}\times\square = \frac{9}{8}$
x $\frac{3}{2}x = \frac{9}{8}$
xi $\frac{3}{2}x = \frac{5}{6}$
xii $\frac{3}{2}x = \frac{7}{8}$

When you were finding the numbers to fit the frames in that last exercise, you were really doing a division process.

Compare $3x = 7 \Leftrightarrow x = 7 \div 3$
with $\frac{3}{4}x = \frac{2}{4} \Leftrightarrow x = \frac{2}{4} \div \frac{3}{4}$
$\Leftrightarrow x = \frac{2}{3}$

For a division process like that, we would find it useful to have a rule. Here it is:

$\frac{2}{4} \div \frac{3}{4} = \frac{2}{4} \times \frac{4}{3} = \frac{2}{3}$

In words: turn the divisor over and multiply it. Try out your skill in the next exercise. The answers to alternate questions are at the end of the section.

Exercise 11 **a**
i $4 \div \frac{1}{4}$
ii $5 \div \frac{1}{4}$
iii $6 \div \frac{1}{2}$
iv $2 \div \frac{1}{3}$
v $4 \div \frac{1}{7}$
vi $8 \div \frac{1}{9}$
vii $4 \div \frac{4}{5}$
viii $5 \div \frac{5}{7}$
ix $6 \div \frac{3}{2}$
x $2 \div \frac{8}{9}$
xi $5 \div \frac{10}{7}$
xii $5 \div \frac{7}{10}$

b Does division always result in a smaller number than those you began with?

c
i $\frac{2}{3} \div \frac{1}{3}$
ii $\frac{2}{3} \div \frac{1}{2}$
iii $\frac{2}{5} \div \frac{1}{4}$
iv $\frac{2}{3} \div \frac{2}{5}$
v $\frac{4}{5} \div \frac{2}{3}$
vi $\frac{7}{9} \div \frac{3}{5}$
vii $\frac{1}{3} \div \frac{2}{3}$
viii $\frac{1}{2} \div \frac{2}{3}$
ix $\frac{1}{4} \div \frac{2}{5}$
x $\frac{2}{5} \div \frac{3}{2}$
xi $\frac{2}{3} \div \frac{4}{5}$
xii $\frac{3}{5} \div \frac{7}{9}$

Now, looking back, we can see we have learnt a lot about fractions. We began with a question: What sort of numbers do we need to solve certain equations when natural numbers will not do? We found that we could define our new numbers as couples; that we could say what we meant by equivalence; that these couples were really ratios; that ratios could be written as common fractions. The new numbers are called *rational numbers.* We have learnt now several rules for doing the arithmetic of rational numbers.

Rule 1 We can find an equivalent number to one we have by multiplying or dividing the top and the bottom by the same number.
Example $\frac{4}{8} = \frac{1}{2} = \frac{8}{16} = \frac{24}{48}$

Rule 2 We can multiply two numbers by multiplying the tops and multiplying the bottoms. The answer may be reduced to its lowest terms afterwards.
Example $\frac{3}{4} \times \frac{5}{9} = \frac{15}{36} = \frac{5}{12}$

Rule 3 The process of division is done by inverting the second number and multiplying.
Example $\frac{3}{8} \div \frac{9}{16} = \frac{3}{8} \times \frac{16}{9} = \frac{48}{72} = \frac{2}{3}$

Rule 4 The reduction to lowest terms can be done earlier on by using Rule 1. This can *only* be done when there is an × sign.

$$\frac{3}{8}\times\frac{16}{9}=\frac{\overset{1}{\cancel{3}}\times\overset{2}{\cancel{16}}}{\underset{1}{\cancel{8}}\times\underset{3}{\cancel{9}}}=\frac{2}{3}$$

This is sometimes called *cancelling*.

Here are some exercises in which you may use any of these rules.

Exercise 12 **a**

i $\frac{4}{7}=\frac{}{21}$
ii $\frac{9}{36}=\frac{1}{}$
iii $\frac{40}{100}=\frac{}{10}$
iv $\frac{3}{16}=\frac{9}{}$
v $\frac{9}{12}=\frac{}{4}$
vi $\frac{48}{72}=\frac{}{3}$
vii $\frac{27}{9}=\frac{}{1}$
viii $\frac{56}{7}=\frac{8}{}$
ix $\frac{3}{10}=\frac{}{100}$
x $\frac{5}{10}=\frac{50}{}$
xi $\frac{3}{8}=\frac{39}{}$
xii $\frac{}{17}=\frac{12}{102}$

b Work out these expressions:

i $\frac{7}{5}\times\frac{4}{3}$
ii $\frac{7}{5}\div\frac{4}{3}$
iii $\frac{5}{7}\times\frac{3}{4}$
iv $\frac{5}{7}\div\frac{3}{4}$
v $\frac{3}{4}\div\frac{5}{7}$
vi $\frac{3}{7}\div\frac{5}{4}$
vii $\frac{3}{7}\times\frac{5}{4}$
viii $\frac{4}{3}\times\frac{6}{8}$
ix $\frac{4}{3}\div\frac{6}{8}$
x $\frac{7}{3}\div\frac{5}{6}$
xi $\frac{5}{6}\div\frac{7}{3}$
xii $\frac{6}{7}\div\frac{9}{14}$

c Solve these equations:

i $3x = 13$
ii $12x = 14$
iii $15x = 35$
iv $4x+5 = 8$
v $3x-7 = 9$
vi $5x+9 = 12$
vii $5x-9 = 12$
viii $12x-4 = 3$
ix $12x-3 = 4$
x $2(4x-5) = 10$
xi $3(2x+4) = 15$
xii $4(3x-1) = 16$

Answers to Exercise 9

c i $\frac{4}{3}$
iii 5
v $\frac{16}{5}$
vii $\frac{20}{9}$
ix $\frac{20}{3}$
xi $\frac{60}{17}$

d i $\frac{10}{3}$
iii $\frac{1}{3}$
v $\frac{8}{9}$

e i $\frac{2}{3}$
iii $\frac{1}{3}$
v $\frac{3}{2}$
vii $\frac{1}{2}$

f i $1\frac{1}{4}$
iii $2\frac{2}{3}$
v $\frac{3}{2}$
vii $\frac{23}{6}$
ix $\frac{39}{8}$
xi $2\frac{2}{7}$

Answers to Exercise 10

a i $\frac{8}{5}$
iii $\frac{13}{4}$
v $\frac{5}{3}$
vii $\frac{3}{2}$
ix $\frac{13}{2}$
xi $\frac{14}{3}$

b i $\frac{1}{2}$
iii $\frac{1}{5}$
v $\frac{2}{3}$
vii $\frac{3}{5}$
ix $\frac{3}{4}$
xi $\frac{5}{9}$

Answers to Exercise 11

a i 16
iii 12
v 28
vii 5
ix 7
xi $\frac{7}{2}$

c i 2
iii $\frac{8}{5}$
v $\frac{6}{5}$
vii $\frac{1}{2}$
ix $\frac{5}{8}$
xi $\frac{5}{6}$

Section 17 **Arithmetic with the rational numbers**

We have already learnt about the multiplication and division of rational numbers. We have not yet seen the more fundamental process of addition.

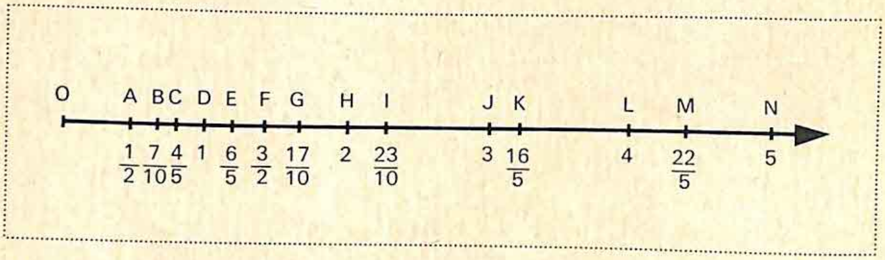
Exercise 1 **a** Can you write out the set $\{x|0 < x < 1; x \text{ is a rational number}\}$? Explain.

b What are the first and the last numbers in that set?

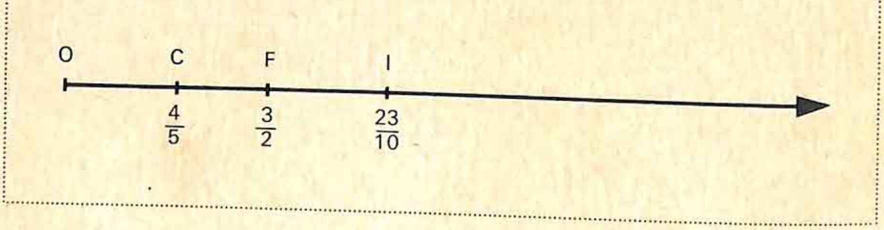
c How many elements do you think there are in the set?

d Could you draw a graph of a finite subset of the set?

Here is a number line. We are going to use the line as a graph of the rational numbers. Every rational number has a corresponding point on the line, but, curiously, there are some points on the line which do not represent any rational number. We shall meet the numbers they do represent later in Book Two.



We have plotted a number of points on the number line and for convenience we have lettered the points. The segment OC represents the size of the number $\frac{4}{5}$ and the distance OF represents the number $\frac{3}{2}$. With a pair of dividers (compasses will do) step off the distance OC from F to the right. You arrive at I. You have really added $\frac{4}{5}$ to $\frac{3}{2}$. You could write $\frac{4}{5}+\frac{3}{2} = \frac{23}{10}$.



Exercise 2 **a** Check the following additions:

i $\frac{1}{2}+\frac{7}{10}=\frac{6}{5}$

ii $\frac{16}{5}+\frac{6}{5}=\frac{22}{5}$

b Work out the following additions. Alternate answers may be found at the end of the section.

i $\frac{1}{2}+\frac{6}{5}$
ii $\frac{3}{2}+\frac{17}{10}$
iii $\frac{23}{10}+\frac{6}{5}$
iv $\frac{4}{5}+\frac{1}{2}$
v $\frac{1}{2}+\frac{4}{5}$
vi $\frac{4}{5}+\frac{6}{5}$
vii $\frac{6}{5}+\frac{7}{10}$
viii $\frac{3}{2}+\frac{23}{10}$
ix $\frac{6}{5}+\frac{23}{10}$

c Try these subtractions by the same method:

i $\frac{6}{5}-\frac{1}{2}$
ii $\frac{3}{2}-\frac{7}{10}$
iii $\frac{22}{5}-\frac{23}{10}$
iv $\frac{16}{5}-\frac{17}{10}$
v $\frac{6}{5}-\frac{4}{5}$
vi $\frac{22}{5}-4$

Now we need to find a rule to help us when we have not a number line beside us.

Look at $\frac{16}{5}+\frac{6}{5}$. This we already know is $16\times\frac{1}{5}+6\times\frac{1}{5}$ which must be $22\times\frac{1}{5}$, and that is $\frac{22}{5}$. We can add when the denominators of the fractions are the same.

Let us try $\frac{4}{5}+\frac{3}{2}$ and rewrite it as $\frac{8}{10}+\frac{15}{10}$ using the equivalence of fractions. This is clearly $\frac{23}{10}$ which is the correct answer. This is our rule: change the fractions to equivalent fractions until the denominators are the same; then add the numerators.

Here are two examples taken from the previous exercise.

A $\frac{4}{5}+\frac{1}{2}=\frac{8}{10}+\frac{5}{10}=\frac{13}{10}$
B $\frac{6}{5}-\frac{1}{2}=\frac{12}{10}-\frac{5}{10}=\frac{7}{10}$

Exercise 3

a b c Work out by the rule all the calculations of Exercise 2 a b c and check your new answers against the previous ones.

d
i $\frac{2}{3}+\frac{3}{5}$
ii $\frac{1}{2}+\frac{1}{5}$
iii $\frac{5}{3}+\frac{1}{8}$
iv $\frac{3}{2}+\frac{4}{3}$
v $\frac{3}{10}+\frac{1}{4}$
vi $\frac{1}{2}+\frac{1}{4}$
vii $\frac{1}{2}+\frac{3}{4}$
viii $\frac{3}{4}+\frac{1}{2}$
ix $\frac{2}{3}+\frac{1}{6}$
x $\frac{1}{3}+\frac{1}{6}$
xi $\frac{1}{3}+\frac{5}{6}$
xii $\frac{1}{10}+\frac{3}{5}$
xiii $\frac{3}{7}+\frac{5}{14}$
xiv $\frac{4}{7}+\frac{5}{21}$
xv $\frac{8}{9}+\frac{4}{15}$

e
i $\frac{2}{3}-\frac{3}{5}$
ii $\frac{1}{2}-\frac{1}{5}$
iii $\frac{5}{3}-\frac{1}{8}$
iv $\frac{3}{2}-\frac{4}{3}$
v $\frac{3}{10}-\frac{1}{4}$
vi $\frac{2}{3}-\frac{1}{6}$
vii $\frac{1}{3}-\frac{1}{6}$
viii $\frac{5}{6}-\frac{1}{3}$
ix $\frac{3}{5}-\frac{1}{10}$
x $\frac{6}{7}-\frac{5}{14}$
xi $\frac{4}{7}-\frac{5}{21}$
xii $\frac{8}{9}-\frac{4}{15}$
xiii $\frac{5}{6}-\frac{2}{15}$
xiv $\frac{7}{8}-\frac{3}{5}$
xv $\frac{17}{12}-\frac{9}{10}$

When we change the denominator of two or more fractions to the same number as we have been doing, we need to find a number into which they will both divide. We need a *common multiple* of the denominators.

For instance, when adding $\frac{5}{6}$ to $\frac{1}{2}$ we need a number into which both 6 and 2 will divide. Clearly 6 will do, so we put them both over 6. If we add $\frac{1}{5}$ to $\frac{1}{3}$, we use a denominator of 15.

If we are adding $\frac{1}{6}$ to $\frac{1}{9}$, we could use $6 \times 9 = 54$, but this is needlessly large. 54 is a common multiple of 6 and 9; but we would be better off with the *least common multiple* which is 18.

Exercise 4 **a** i Complete the following: $\frac{1}{2}+\frac{1}{3}=\frac{}{12}+\frac{}{12}=\frac{}{12}=\frac{}{6}$

ii Repeat the calculation using the least common multiple of 2 and 3.

b What are the least common multiples of these pairs of numbers? (Alternate answers are at the end of the section.)

i 5 and 6
ii 4 and 6
iii 3 and 8
iv 6 and 15
v 12 and 16
vi 10 and 12
vii 8 and 10
viii 8 and 6
ix 14 and 21

c Use the information you have just discovered to work out the following. Answers to the even numbered ones are at the end of the section.

i $\frac{1}{5}+\frac{1}{6}$
ii $\frac{1}{4}+\frac{1}{6}$
iii $\frac{2}{3}+\frac{3}{8}$
iv $\frac{5}{6}-\frac{2}{15}$
v $\frac{5}{12}-\frac{1}{16}$
vi $\frac{3}{10}+\frac{5}{12}$
vii $\frac{7}{8}-\frac{3}{10}$
viii $\frac{5}{8}-\frac{1}{6}$
ix $\frac{11}{14}+\frac{5}{21}$

Exercise 5 The upper number in a fraction is called the *numerator* and the lower number is called the *denominator*.

a i Find all the proper common fractions (reduced to their lowest terms) whose denominators are 7 or less. There are 17 of them. Arrange them in order of size, beginning with the smallest. This is the *Farey series of order 7*.

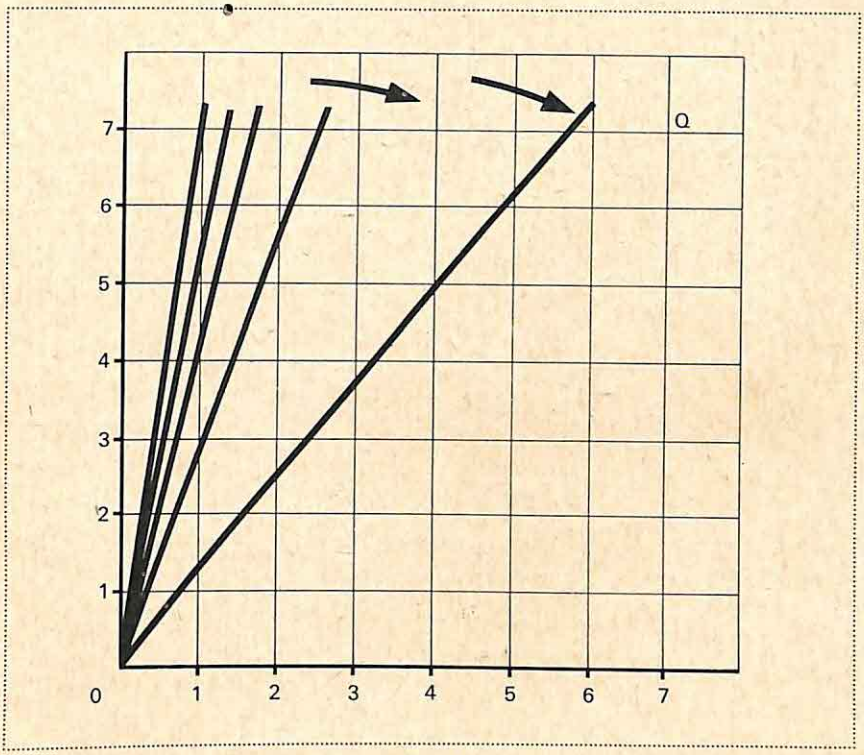
ii Where is $\frac{1}{2}$ in the series?

iii Find a pair of numbers in the series equidistant from $\frac{1}{2}$ on either side of it and add them. Now find another pair and add them. Repeat this for all the pairs.

iv Look at any any two successive fractions. Find their difference. Now multiply their denominators and find the reciprocal of this number. Does this always work?

iv Draw this lattice on graph paper.

Regard the numbers along the 'horizontal' as the numerators and the 'vertical' numbers as the denominators. Place a ruler on your lattice in the 'vertical' position. Slowly rotate the ruler clockwise. The ruler passes through intersections on the lattice in turn. Write down the numbers as they appear. If there are two on one line, take only the one nearest to the origin (why?). Carry on until you reach the position OQ. What do you notice about the sequence you have written down?



v Why do we stop at the diagonal? What happens if we continue?
vi Why do we ignore the points outside the 7 by 7 lattice?

b Repeat all this to find:

i how many fractions there are in the Farey series orders 1, 2, 3, 4, 5, 6;
ii whether the arithmetic properties we discussed above always hold;
iii and whether the graph method always works.

Does nature work in fractions? Obviously, we have such things as whole numbers in the world: we can see one hill, five mountains, two rivers if we look in the right places. Nature clearly works with whole numbers, which are also called *natural numbers*. The number *zero* also occurs in nature. It is the number of five-legged cows or horses with horns. But we would be very surprised if a farmer said he had $24\frac{1}{2}$ cows or if a school class was of $31\frac{3}{4}$ pupils. On the other hand we can have half a cake, $\frac{2}{3}$ of a bucket of water or $\frac{1}{4}$ of a field. Fractions do apply to some things and not others.

Exercise 6 By considering the question 'Does half of – mean anything?' in each case, arrange the following in two lists. Into list A put those things to which fractions do apply and into list B those to which they do not.

Ball, dog, pudding, bottle of ink; onion, corner, 10, notebook, tin of paint, roll of wallpaper, dinner.

You should have found that it is not easy to decide in some cases. The use to which the object is to be put is part of the problem; the context is important. I may have half a tin of paint at home, but I cannot buy half a tin in a shop. The same applies to wallpaper. You can eat half your dinner, but the waiter will not refund on the other half. In no circumstances however can you turn half a corner.

Whenever we use fractions or give answers in fractions, we must always remember to consider whether fractions can mean anything.

Exercise 7 **a** Puppies cost £2 each. I have £5. How many can I buy?

b I can paint six doors with a tin of paint. How much shall I need for nine doors? How much shall I have to buy?

c Notepaper costs 2/– for a packet of a 100 sheets. How many sheets can I buy for 9/–?

d A car will travel 40 miles to a gallon of petrol. How many gallons will be used to do a journey of 60 miles? If the tank were empty at the beginning of the journey, how few gallons could we manage with?

Exercise 8 Say whether the following are true or false:

a
i $1\frac{3}{4} = \frac{7}{4}$
ii $1\frac{1}{2} = \frac{5}{4}$
iii $\frac{3}{2} = 1\frac{1}{2}$
iv $\frac{9}{4} = 2\frac{1}{4}$
v $\frac{5}{3} = 1\frac{2}{3}$
vi $\frac{7}{3} = 2\frac{2}{3}$
vii $\frac{1}{2}+\frac{4}{2} = 2\frac{1}{2}$
viii $\frac{7}{5} = 1\frac{1}{2}$
ix $\frac{2}{5}+\frac{4}{5} = 1\frac{1}{5}$
x $2\frac{2}{5} = \frac{9}{5}+\frac{3}{5}$
xi $3\times\frac{1}{5}+\frac{1}{5} = 1$
xii $3\times\frac{1}{5}+3\times\frac{1}{5} = 1$

Rewrite the following in a different way:

b i $\frac{7}{4}$ ii $1\frac{3}{8}$ iii $2\frac{1}{8}$ iv $\frac{16}{5}$ v $\frac{14}{9}$

c Comment on each of the following sets of expressions:
i $23;\ 2\times 3;\ 2\times 10+3$
ii $\frac{1}{3}\times 2;\ \frac{2}{3};\ 2\div 3;\ 2\times\frac{1}{3};\ \frac{1}{3}$ of 2
iii $2\frac{1}{3};\ 2+\frac{1}{3}$

d We often say 'of' in connexion with fractions. Work out each of the following:
i $\frac{1}{2}$ of 12
ii $\frac{1}{2}$ of $\frac{6}{7}$
iii $\frac{1}{4}$ of $\frac{4}{5}$
iv $\frac{3}{4}$ of $\frac{8}{9}$
v $\frac{7}{8}$ of $\frac{8}{11}$
vi $\frac{3}{8}$ of $\frac{1}{4}$
vii $\frac{5}{8}$ of $\frac{7}{8}$
viii $\frac{3}{8}$ of $\frac{7}{8}$
ix $\frac{4}{5}$ of $\frac{15}{28}$

Answers to Exercise 2

b i $\frac{17}{10}$ iii $\frac{7}{2}$ v $\frac{13}{10}$ vii $\frac{19}{10}$

c i $\frac{7}{10}$ iii $\frac{21}{10}$ v $\frac{2}{5}$

Answers to Exercise 4

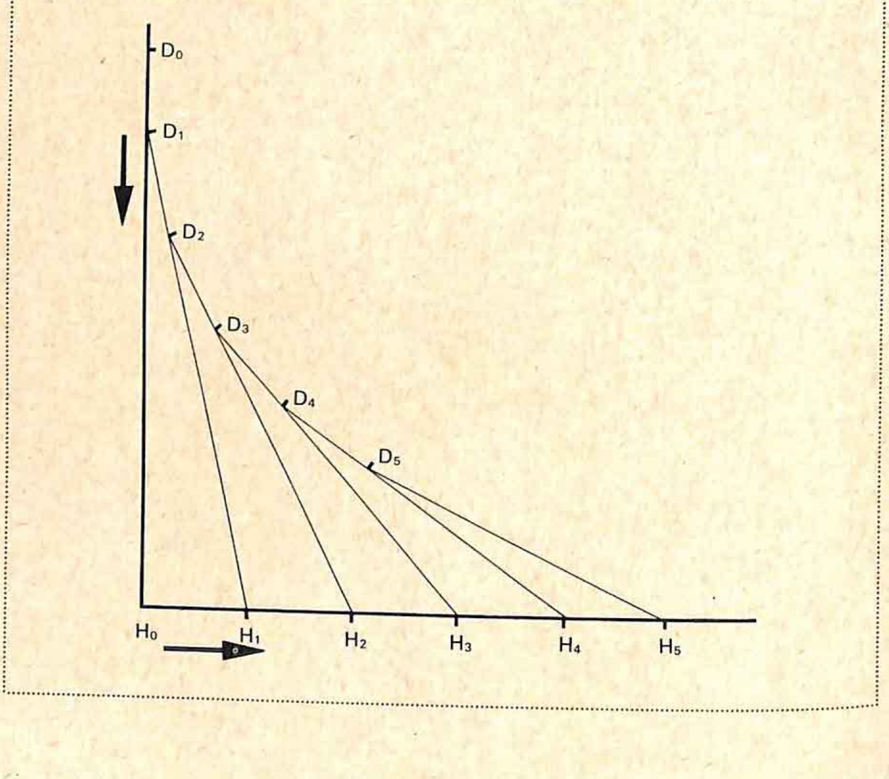
b i 30 iii 24 v 48 vii 40

c ii $\frac{5}{12}$ iv $\frac{7}{10}$ vi $\frac{43}{60}$ viii $\frac{11}{24}$

Section 18 **Drawing curves**

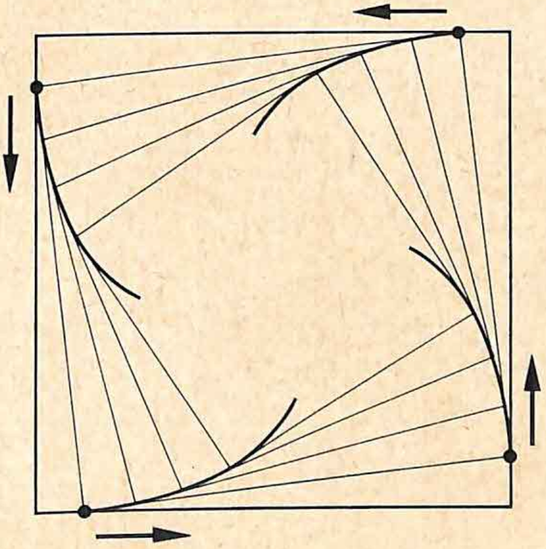
Here are some curves for you to draw. You will find that you have to draw very fine lines with a sharp pencil if the results are to be good. You will find the three in Exercises 2 and 3 are best done on graph paper.

Exercise 1 **a** A dog sees a hare a good distance away. He runs straight towards the hare, but the hare, seeing the dog, runs along the edge of the field at right angles to the line along which the dog begins running. They run at the same speed. As the hare moves, the dog always runs towards him. Does the dog follow a straight path all the time? We have begun in the diagram to draw the path which the dog follows. After one second the dog is at D_1 and the hare at H_1; a second later the dog is at D_2 (on a line drawn from D_1 to H_1), but the hare has moved on to H_2.
Copy and continue this diagram.

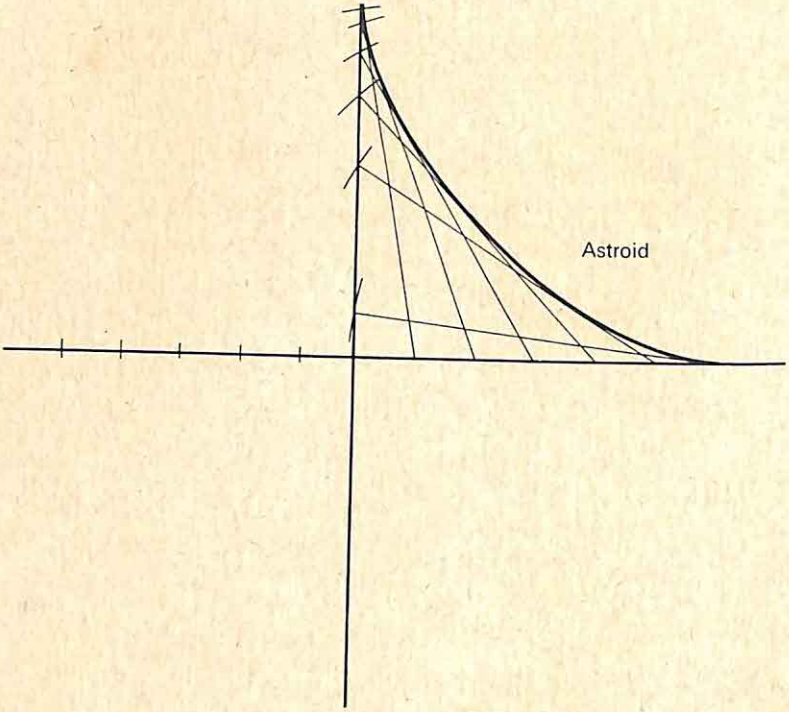


Does your finished drawing look like a curve? Is it really a curve? If the dog always moved directly towards the hare would he for instance actually run along the straight line D_0D_1? Would he be very far off this line?

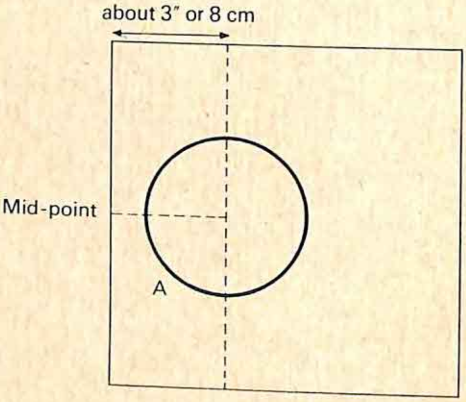
b Here is a similar problem but one which gives a different result. We have this time four dogs at the four corners of a square field. Each dog is facing in the direction of the arrow at his corner and he can only see one other dog. The dogs are released, and as dogs will, they run after one another, but they all run at the same speed. We have begun to draw their paths, by the same method as (a) above. Finish it for yourself. What happens in the end?



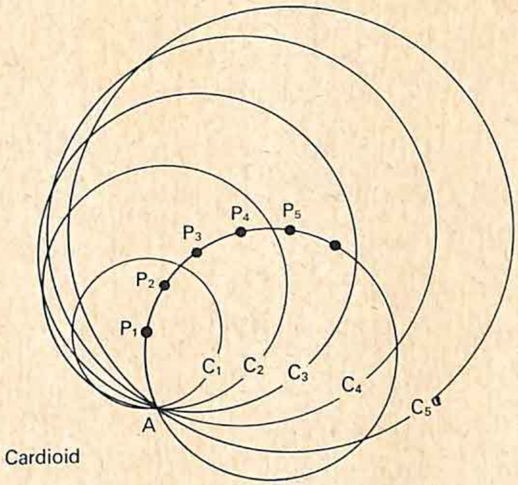
Exercise 2 a Choose a piece of paper at least 7 in. or 18 cm. square. Draw two lines at right angles to make a cross in the centre of your paper. Mark off one of the lines in $\frac{1}{4}$ in. or 1 cm. units. Take a radius of about 3 in. or 8 cm. on your compasses. From each of the points you marked on one line draw an arc to cut the other line twice. Draw these arcs very lightly so that you can only just see them. Now join each intersection to its centre. Label your drawing 'An astroid'.



b Again choose a piece of paper rather bigger than 7 in. or 18 cm. square.

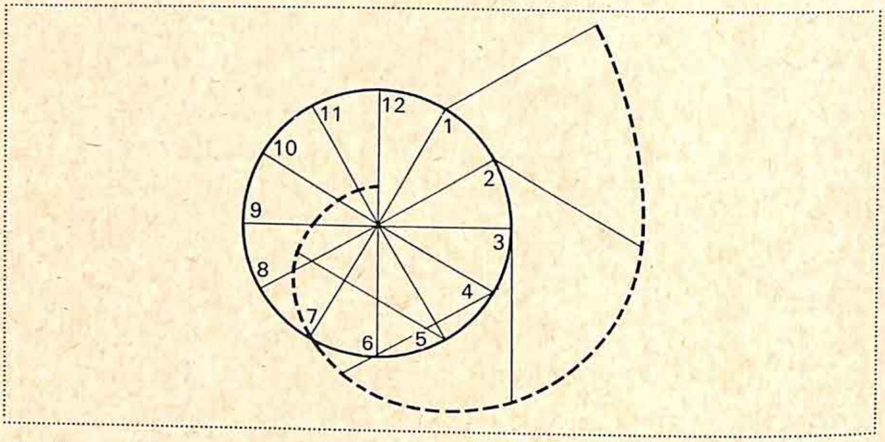


In the position shown on the diagram, draw a circle radius about $1\frac{1}{2}$ in. or 3·5 cm. Mark the point A. Now at intervals round the circumference of the circle mark points P_1, P_2, P_3, etc. The drawing will look better if you put these at equal intervals. You will need about eighteen of these points (including A) all round the circumference. Now with centre P_1 and radius P_1A draw a circle. Next with centre P_2 and radius P_2A draw a circle. Carry on round the circle in this way. The curve which surrounds your whole picture is called a *cardioid*. Label your drawing.



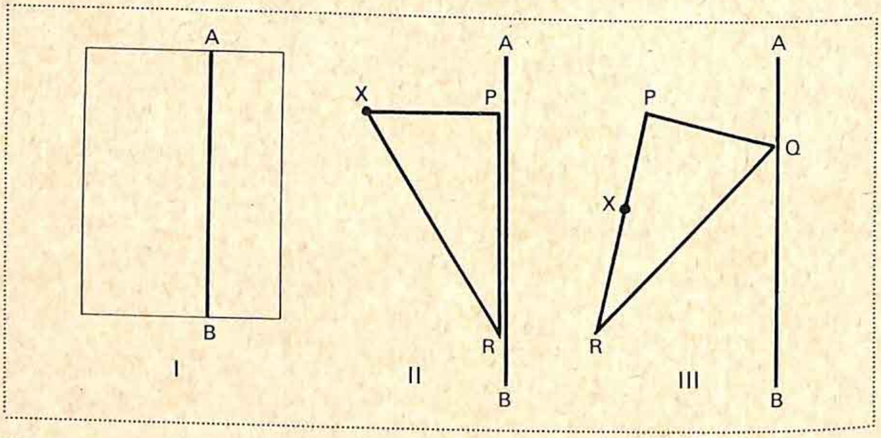
In all the drawings that you have done so far in this section, the final curve was not put in by you—it just appeared as if by magic. In three of the examples you were not even drawing curves, but you still got a curve at the end. Obviously, your curve was really a series of straight line segments, but these were very short and could be made shorter still if you drew more of them. And, as we say, in the *limit* the result is a curve. When a curve grows in this way, it is said to be the *envelope* of the lines you have drawn. Can you see why?

Exercise 3 Here is a curve which is not an envelope. You will have to complete the curve by joining points with a smooth line. In the centre of your paper, draw a circle of radius about $1\frac{1}{2}$ in. or 4 cms. Draw in radiuses like the hours on a clock and number them 1 to 12. From each of the twelve points on the circumference you will be drawing a line segment of the same length, say 2 in. or 5 cm. From 1 you will draw your segment parallel to the radius to 2; from 2 a segment parallel to the radius to 4; from 3 a segment parallel to the radius to 6; and so on all round the circle. The ends of the segments are then to be joined by a smooth curve to form a *limaçon*. The diagram shows a little less than half the curve. You may find that some of the points are too far apart. If so you can put in intermediate points. This method of drawing a limaçon was used by the German artist Albrecht Dürer in the sixteenth century.



Exercise 4 To draw this locus you will need a 60° set square, a sheet of drawing paper, a ruler and a sharp pencil. The width of the paper must be at least twice the size of the shortest side of your set square; its length must be at least four times the shortest side.

Place your paper as in Figure I and draw a line segment AB about one third of the way across as shown.



Now place your set square as in Figure II and mark the point X. Be sure you have the set square the right way up.

Now rearrange your set square as in Figure III so that the second longest side just goes through point X and the vertex Q lies on AB. Mark as many positions of P as you can fairly close together. P can be on the other side of AB. Can you see how? This subset of the plane is called a *right strophoid*.

Section 19 **Using numbers**

What time is it? How do you *know*? What do we mean when we say 'It's 5 o'clock'? Who decides what moment is to be called midnight? How are we to settle the length of a day?

Even when that is decided, how are we going to divide the day into parts? These questions have been of great concern ever since man began to be civilized.

For many centuries men tried to use natural phenomena or events to give them a means of measuring time. After all the only thing we can all be certain about without any help is whether something happened before or after something else. The first method was the use of *solar time* or the time given by the sun. This is the time given by the sundial. The days are not all equal in length however. Since the seventeenth century when clocks became common, the sundial has dropped out of use. We still need to know how fast to make our clocks go. Astronomers have spent a good deal of effort in trying to do this exactly. They have invented *sidereal* time, *universal* time and lately *ephemeris* time. They are now able to give the times of astronomical events very accurately indeed.

Another way of measuring time is to use a man-made instrument. The instrument must be able to show some kind of rhythm or regularity. One such instrument was the water-clock or *clepsydra*. Another was the lamp or candle which takes a precise time to burn out. The hour-glass was used until quite recent times, in fact some old churches still have pulpit glasses which were used in the eighteenth century to time the sermon!

There is a story that a young student at the University of Pisa in Italy who, in his first year of study in 1581, went to a service in the cathedral. There he watched a lamp swinging gently in a breeze during the sermon. He noticed one important thing about the lamp and as a result he discovered the principle of the pendulum. The student's name was Galileo Galilei, later to be famous for his research with a telescope which he made himself. The addition of a counting mechanism (or *escapement*) to the pendulum by Christian Huygens in 1656 gave us a clock mechanism which is not so very different from that used today.

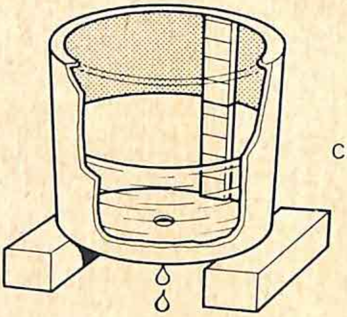
Exercise 1
Things to find out

1 Find out all you can about solar time and sidereal time.
2 Find out why the time is different in different longitudes, and find out about the international time zones used in telecommunications.

3 Find out how ancient peoples used to measure and divide the day. (Babylonians, Egyptians, Jews, Romans, for instance.)
4 Find out about the watches (not clocks!) used at sea.
5 Find out all you can about old instruments for measuring time. (For instance the 'Chinese' alarm clock.)
6 Find out about Galileo's work on the pendulum.
7 Find out why men decided on 24 hours to a day, 60 minutes to an hour, and 60 seconds to a minute.

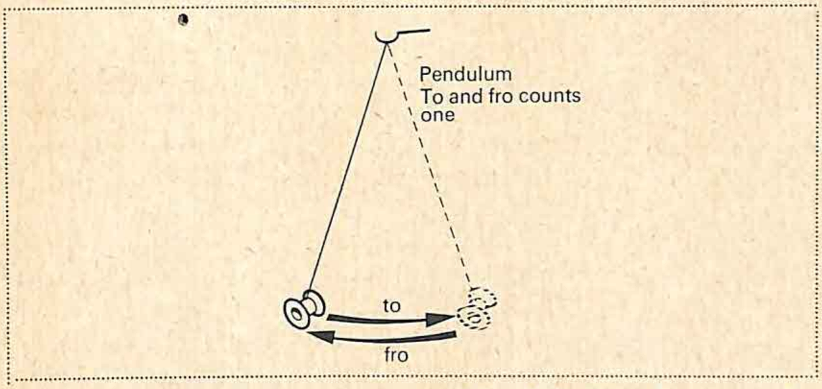
Exercise 2
Things to do

1 Make a clepsydra. Take a large tin (a $1\frac{1}{2}$ lb. Nescafé tin is ideal) and punch a hole in the centre of the base with a nail. The hole is best punched from the inside. Wedge a smooth unpainted strip of wood upright inside the tin. Fill the tin with water and stand it outside on two bricks so that it can drip steadily. If it does not drip steadily, enlarge the hole slightly. Allow the water to drip out and after suitable intervals of time (say hours or half-hours) mark the level of the water on the wood. (A ball pen is useful for this since the marks will not wash off.) You have now made and calibrated a clepsydra, and the units you have used are the one we all use for our mechanical clocks today.



Clepsydra

2 Carry out some experiments like Galileo's. Hang a cotton reel from a hook by about a yard of thread to make a pendulum. Draw the cotton reel about a foot to the side and let go. Count or tap your foot (quietly!) each time your pendulum returns to its original position. Now draw the pendulum further to the side and let go. Does it travel faster? Count or tap again. Were you counting (a) more quickly, (b) at the same speed or (c) more slowly? Now try the experiment with a shorter thread. What difference does this make to the time of swing?



How long does it take it to swing to and fro if the thread is just under 10 in. from the hook to the centre of the cotton reel? (The best way is to time thirty swings and then divide the time by 30; a rough answer will do.)

If you wanted to write down your age, you would not do it in days. Why not? Normally you would use a larger unit and write it in years. How many days are there in a year? If you said 365, think again. If we had a year of exactly 365 days, in time the calendar would get out of phase with the sun and we should have snow in August and Christmas cake in a heat wave. This would take a long time though. We put the calendar nearly right by putting in and leaving out leap years. Here are the rules:

Every fourth year shall be a leap year except the last year of each century which shall be a leap year only if the number of the century is divisible by 4. On the other hand the year 4000 and all its multiples are not to be leap years. Here are some examples:

1964 was a leap year because 1964 divides by 4 and the year was not the end of a century.

1900 was not a leap year because it was the end of a century and 19 does not divide by 4.

2000 will be leap year because it is the end of a century and 20 divides by 4.

8000 will not be a leap year because it is a multiple of 4000.

This complicated adjustment of the calendar will keep the calendar so nearly 'right' that the end of each year will not vary from its present place by more than a day for the next 200 centuries. After that, the correction is a little more inaccurate!

Exercise 3

Things to think about and find out

1 Find out the precise length of a year in days, hours, minutes and seconds. (This is called the *mean solar year.*)

2 Does it mean anything to talk about the number of seconds in a calendar year? Explain.

3 The lunar month is 28 days. Why are the months of the year other numbers of days? Find out all you can about this.

4 Find out how to calculate the date of Easter. The instructions are at the beginning of the Book of Common Prayer.

5 Find out what is meant by the Gregorian Calendar or the New Style.

6 What arrangements were made in the U.K. in connexion with the calendar by the Calendar (New Style) Act of 1751?

7 Find out all you can about ancient calendars.

8 How many days of holiday would you have if you finished school at 4 p.m. on 24th July and reassemble on the morning of 5th September? (Include Saturdays and Sundays.)

9 I pay for my Sunday newspapers a month in advance and I leave the money out for the paper-boy. How many times a year do I have to leave out enough money for five Sundays' papers?

10 A girl gets her pocket money monthly in advance. She is given an amount which depends on the number of Fridays in the month. Does she get her five-week money in the same months as I pay extra to the paper boy?

11 What was the date a fortnight before 14th September, 1752?

12 Julius Caesar died in the year 44 B.C. In the year AD 61 the Iceni under Boadicea revolted. How many years were there between these events?

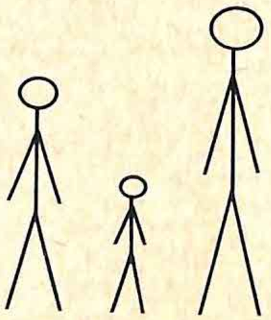
13 A train leaves London at 10.10 a.m. and arrives in Hull at 3.44 p.m. How much time did the journey take?

14 If I drive along the motorway M1 from the London end to Crick, it will take me 67 minutes provided I do not stop. If I spend a quarter of an hour getting a cup of coffee on the way, can I leave the motorway by 8.15 a.m. having started the journey at 6.55 a.m.?

15 The times of continental trains and times in H.M. armed services are given in terms of the 24-hr. clock. What is 15.30 hrs. in 'ordinary' clock time? Convert 'half-past eight in the evening' to 24-hr. clock time

16 What do the letters a.m. and p.m. mean?

17 What is the International Date Line and where is it?



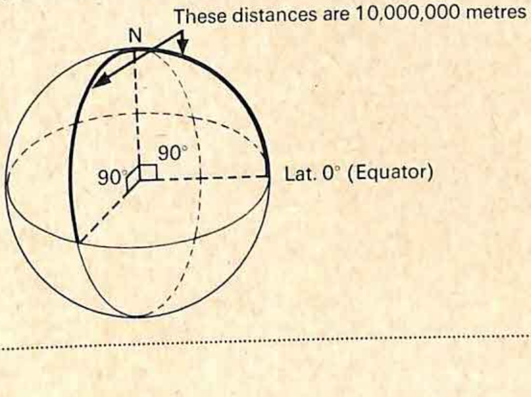
Man, boy and giant

Look up the story of Goliath in your bible: I Samuel 17, 4. Can you say straight away how much bigger than you Goliath was? Suppose I measure the width of this book with my thumb, using the distance from the knuckle to the tip. I find it is a measure of about five of these units. Do the same yourself. Did you get 5 as the answer? Is your book the same size as mine? What is different then?

Of course if we want to measure things fairly accurately and then exchange information we must all use the same units. This was understood in A.D. 965 when King Edgar proclaimed that the standard foot should be the same for the whole land,' such as is observed at London and Winchester', where they kept a standard foot and a standard yard.

Modern standards are to be found at the Board of Trade and copies in other secure places.

Our system of units in Britain is as you know very ancient and very complicated. It is soon to change.

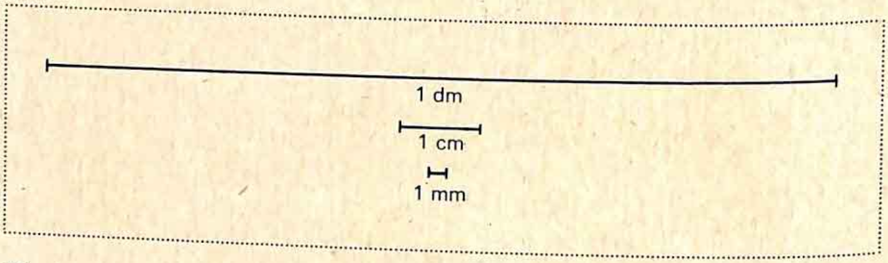


In 1799 the French National Assembly enacted that, from 1801, they would use a new system of measures. The system is now called the *metric system* and the basic unit of length is the *metre*. The system was devised in this way: assume that the earth is a perfect sphere; divide the distance from the equator to the north pole by the shortest surface route into 10,000,000 parts; each one of these is a metre. One metre then works out to be just less than 39·4 in. or a little more than a yard. They established a system by multiplying and dividing the metre by 10. Here are the units in a table.

Approximate English equivalent	Metric units of length	Abbreviation
0·62 miles	1000 metres = 1 kilometre	km.
	100 metres = 1 hectometre	Hm.
	10 metres = 1 decametre	Dm.
39·4 ins.	1 metre	m.
	0·1 metres = 1 decimetre	dm.
0·39 ins.	0·01 metres = 1 centimetre	cm.
	0·001 metres = 1 millimetre	mm.

Approximate metric equivalents of English units

1 in.	2·54 cm.
1 ft.	30·5 cm.
1 yd.	0·91 m.
1 mi.	1·6 km.



Exercise 4 Work out:

a
- i How many decametres there are in a kilometre.
- ii How many cm. there are in a Dm.
- iii 1 mm. as a decimal fraction of 1 Hm.
- iv 1 cm. as a decimal fraction of 1 km.
- v 10 mm as a decimal fraction of a metre.
- vi How many cm. there are in 3·5 Dm.

b i How many mm. there are in 3·4 cm.
ii How many mm. there are in 3·4 m.
iii How many cm. there are in 3·4 Dm.
iv How many cm. there are in 3·4 Hm.
v How many m. there are in 3·4 Dm.
vi How many m. there are in 3·4 km.

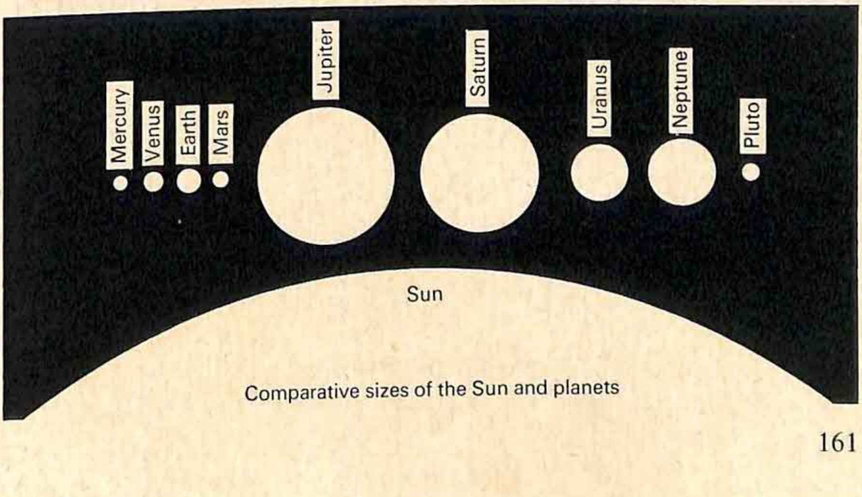
c i What decimal fraction of 1 m. is 47·6 cm.?
ii What decimal fraction of 1 km. is 47·6 cm?
iii What decimal fraction of 1 Hm. is 47·6 mm.?
iv What decimal fraction of 1 km. is 47·6 mm.?

d Express in m.
i 1·25 km.
ii 0·85 km.
iii 0·06 km.
iv 1250 cm.
v 55 cm.
vi 32 Hm.
vii 7·6 Dm.
viii 0·76 dm.
ix 0·543 km.
x 543 mm.
xi 54·3 dm.
xii 54·3 Dm.

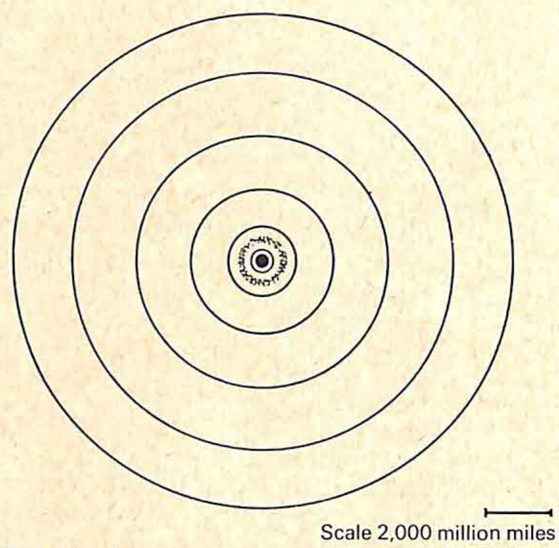
e i About how many yards is 12 m.?
ii About how many feet is 340 Hm.?
iii About how many inches is 5 m.?
iv About how many miles is 8 km.?
v About how many inches is 260 cm?
vi About how many metres is $1\frac{1}{4}$ yd.?
vii About how many centimetres is 5 yd.?
viii About how many metres is 9 yd.?

Exercise 5 Say which of the following are reasonable dimensions for the objects described:

a A pocket diary 29 cm. wide
b A desk 100 cm. long
c A man 1·8 m. high
d A man's hand 20 mm. long
e A pencil 1·4 dm. long



Comparative sizes of the Sun and planets



The orbits of the planets are nearly circular.
The dotted band is the orbit of the minor planets of which there are over 2,000. Inside this lie the orbits of Mars, Earth, Venus and Mercury and, at the centre, the Sun.

Astronomers need some very large units of length; inches, feet and even miles are very little use to them. Our nearest astronomical neighbour, the moon, is 250,000 miles away. From the earth to the sun is 93,000,000 miles. This is a very long way: we find such big numbers very hard to grasp.

The distance from the earth to the sun is small compared with other distances in the universe. Astronomers sometimes use a unit called an *astronomical unit* equal to the distance from the earth to the sun. Light travels at about 186,000 miles each second. The *distance* it travels in one year is often called a *light year*. Astronomers also use a unit called a *parsec* which is about 3·3 light years. Even so they have to talk about distances of many millions of parsecs!

Exercise 6 **a** Assume the speed of sound to be about 800 m.p.h. How long (in days) would it take at this speed to reach the moon?

b How long would it take at the speed of sound to travel a distance of 1 astronomical unit?

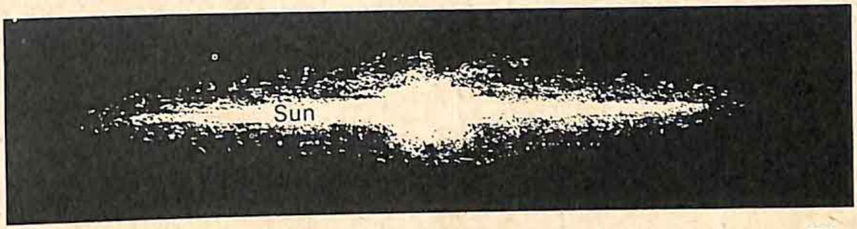
c The planet Pluto is about 39·5 astronomical units from the sun. How long would it take to travel that distance at 1000 m.p.h.?

d When Mars is at its closest to Earth it is more than 0·5 astronomical units away. How long would it take to get there at 25,000 m.p.h.?

e At its furthest away Mars is a little more than 2·5 astronomical units. How many million miles is this?

f How long would it take light to travel from the Sun to Earth?

g



This sketch is an edge-on view of the galaxy of which we are part. The stars we see in the Milky Way are in this galaxy. The distance across the galaxy is about 30,000 parsecs. How many light years is this?

h The central bulge of the galaxy is about 5000 parsecs thick. How many light years is this?

i Our nearest neighbour in space is a star called α-Centauri. It lies at a distance of 4·3 light years. If a space ship travelled at 186 miles per second, how long would it take to get there?

Exercise 7 **a** On land, both in Great Britain and the U.S.A., the mile is a *statute mile* of 1760 yds. How many feet is this?

b At sea a mile is called a nautical mile. In this country we use a standard called the *Admiralty mile* of 6080 ft. Which is greater, the statute mile or the Admiralty mile?

c How many km. are approximately equal to 5 miles?

d The *chain* is a unit still used by surveyors. It is the length of a chain of linked metal rods called Gunter's chain, which has 100 links. A chain is 66 ft. long. How many chains are there to the mile?

e How long is each link of a Gunter's chain?

f For certain purposes in surveying, a bigger unit of 10 chains is used. It is called a *furlong*. How many inches are there in a furlong?

g How many furlongs are there to 1 mile?

h Long ago, the reach of a grown man between the tips of his middle fingers when his arms were outstretched sideways was used as a unit. It came to be called a *fathom* and was standardized at 6 ft. How many fathoms is 500 ft.?

i The water at the Dogger Bank is between 10 and 20 fathoms deep. What is this in feet?

j A region of the North Sea to the west of the Skagerrak is between 500 and 5000 ft. deep. Would an anchor chain 800 fathoms long be certain to reach the bottom?

k 'Full fathom five, thy father lies' sings Ariel in Shakespeare's *The Tempest*. In how many inches of water did Ferdinand's father drown?

l A schoolboy is used to running in a 3-mile cross-country race. He is joining with some French friends at Easter in their 5000 m. race. Will he find it a longer or a shorter distance? How many yards different will it be?

m An owner of a market garden wishes to fence off his land from the road. He finds that he can buy fencing in 20 ft. lengths and he needs a post at every join between them. If his frontage to the road is exactly $\frac{1}{4}$ m. long, how many sections of fencing and how many posts will he need?

Exercise 8 **a** Look up the word *avoirdupois* in a good dictionary.

b Compile a table of ordinary weights including ounces (oz.), pounds (lb.), stones (st.), quarters (qrs.), hundred-weights (cwt.), and tons.

c There is another system of weights used for gems and precious metals. It is called *Troy weight*. Find out what you can about it. (A good dictionary might help you here.)

d How many lb. are there in 1 cwt.?

e How many lb. are there in 1 ton?

f In England a man gives his weight in stones and pounds, while in the U.S.A. a man would know his weight in pounds. Archibald Smith-Jones and Hank Miller are discussing their weights. Smith-Jones knows his is 13 st. 3 lb. while Miller weighs 160 lb. Which is heavier?

g Construct a conversion table for weights between 7 and 17 st. to show how they can be converted to lb. and vice versa.

h In town in East Yorkshire, potatoes are bought by parts of a stone, but in London by pounds. A housewife from London living in Hull knows she needs at least 4 lb. of potatoes. Will $\frac{1}{4}$ st. be enough?

i A coalman has inaccurate scales. His sacks should be $1\frac{1}{4}$ cwt. but are 1 lb. light. What fraction of a sack does he gain in a sale of 5 tons?

j A modern combustion stove is about 60 per cent efficient. That means that about 60 per cent of the heat in the fuel gets into the room. How many cwt. of fuel would you have to buy to get the equivalent of all the heat from a ton?

At the same time that the French invented a new system of length measurement, they also introduced a new system of weights. The basic unit this time is the *gramme* and the system is built up by repeatedly multiplying or dividing it by 10. Here is a table.

Approximate English equivalent	Metric units of weight	Abbreviation
Just less than 1 ton	1000 kg = 1 metric ton or 1 tonne	
2·2 lbs.	1000 gm = 1 kilogram	kg.
	100 gm = 1 hectogram	Hg.
	10 gm = 1 dekagram	Dg.
0·035 oz.	1 gramme	gm.
	0·1 gm. = 1 decigram	dg.
	0·01 gm. = 1 centigram	cg.
	0·001 gm. = 1 milligram	mg.

Approximate metric equivalent
1 lb. is about 0·45 kg.

Exercise 9 **a** On the continent the word kilogram is often shortened to *kilo* (or *chilo* in Italy).

i Is half a kilo of potatoes more or less than a pound?

ii Is the difference about

A. $\frac{1}{4}$ oz. B. 4 oz. C. 1 oz. D. $1\frac{1}{2}$ oz.?

b What is the difference between the ton and the tonne?

c Write the following as decimals of a gm.:

i 45 cg. ii 64 mg. iii 132 mg. iv 245 dg. v 227 cg.

d Convert to gm.:

i 5·34 kg. ii 78·471 kg. iii 14·3 Hg. iv 12·9 Dg.

e Convert to kg.:

i 57 gm. ii 2678 gm. iii 94 Dg.

iv 4·5 tonnes v 45,392 cg. vi 1 mg.

f Does a quarter-pound bar of chocolate weigh more or less than 1 Hg.?

g I have just looked at three imported foodstuffs and seen the following equivalents. Are they correct?

i Tomato Ketchup 20 oz. (575 gm.)
ii Caviar 2 oz. (56 gm.)
iii Smoked sprats 6 oz. (160 gm.)

British money is called *sterling* currency. The pound (£) in Great Britain is called the *pound sterling* while the pounds which used to be used in Australia were called *Australian pounds* (£A). The word sterling is also used to distinguish pound (money) from pound (weight). In February 1966, the Australian currency was made decimal. The old 10*s.* was changed to 1 dollar ($) and divided into 100 cents.

Exercise 10 **a** Is the new Australian cent of greater or smaller value than the old 1d.?

b The florin which we use today was introduced in 1848. What fraction of £1 is it and why do you think it was introduced?

c A gold coin called a *florin* was in use in the reign of Edward III. Its value was $\frac{1}{3}$ of £1. How much is this in shillings and pence?

d A boy saves 3d. pieces and has collected 200. How many more does he need to have £3?

e A mink coat is marked at 275 gns. in a sale. How much does it cost in £s.?

f Many of the questions you have been doing in this section have been about British weights and measures. Why will you not need this information after 1970?

There are quick ways of calculating some costs in British coinage. The *dozen rate* is one. For instance 12 articles at 6d. each cost 6*s.* and 12 at 5½d. each cost 5s. 6d.

Exercise 11 **a** Explain why the dozen-rate method works.

b Work out these quickly:

i Two dozen 2½d. stamps
ii 18 fourpenny stamps
iii 11 boxes of matches at 5d. each
iv 10 protractors at 9½d. each
v 20 books costing 15*s.* each
vi 30 books at 14*s.* 6d.
vii 30 books at 6*s.* 8d. each
viii 20 books at 13*s.* 4d. each.

Table of prefixes used in the metric system.

tera-	1,000,000,000,000
giga-	1,000,000,000
mega-	1,000,000
kilo-	1,000
hecto-	100
deca-	10
deci-	0·1
centi-	0·01
milli-	0·001
micro-	0·000001
nano-	0·000000001
pico-	0·000000000001

Section 20 **About paths**

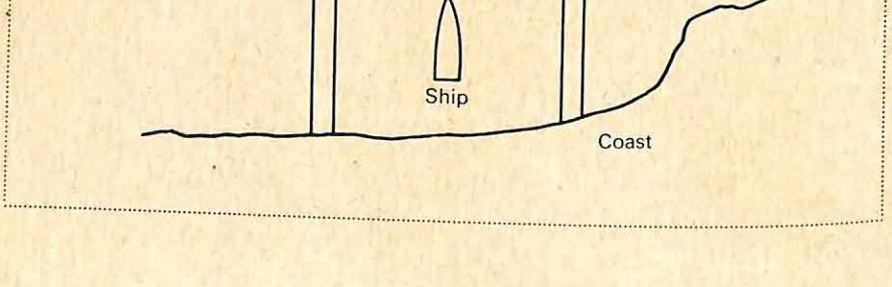
Exercise 1 **a** If a man goes by helicopter from Truro to Launceston 'as the crow flies', the distance he travels is 40 miles. What does this mean? Find these places on a road map. Suppose he went by car by way of Lostwithiel and Liskeard, how far would he travel? Could he find another route shorter than his helicopter route?

b Suppose he wants to go from Libreville to Stanleyville (both in Africa on the Equator) by the shortest route, what path does he have to follow? Is this the same kind of path as the helicopter route from Truro to Launceston? Explain your answer. Are we right in regarding Truro and Launceston as being on a flat surface and Libreville and Stanleyville as being on a sphere? Why?

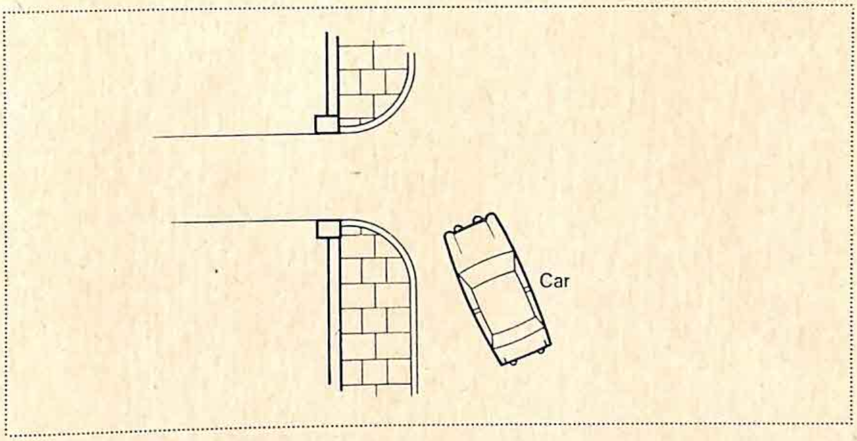
c A goat is tethered to a stake in the middle of a large field. If the goat extends the rope as far as he can, what path does he have to follow?

d If a learner driver begins by carefully 'following' the kerb, exactly what path does he follow when the kerb is absolutely straight? What is the path when the car is following the kerb all the way round a circular traffic island? (You may find that a metal box moved along a ruler, and round a circular tin will help you to see the answers.)

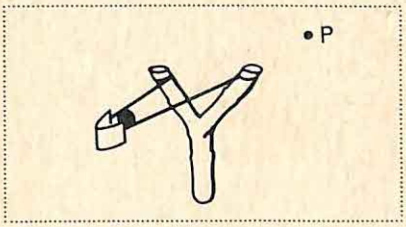
e The sketch shows a narrow harbour mouth. On the ends of the harbour walls there are two look-out towers. What is the exact path between the towers most likely to be followed by a ship leaving the harbour?



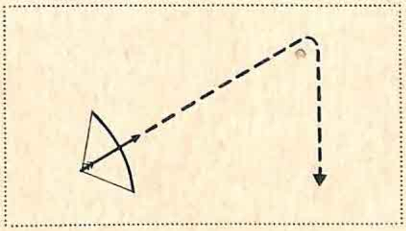
f The sketch shows a narrow gate, entrance to a garage. The gate posts are concrete and will easily damage a car. There is only just enough room for a car to go through. Make a sketch to show the path a car will have to take if it is to get through unscratched.



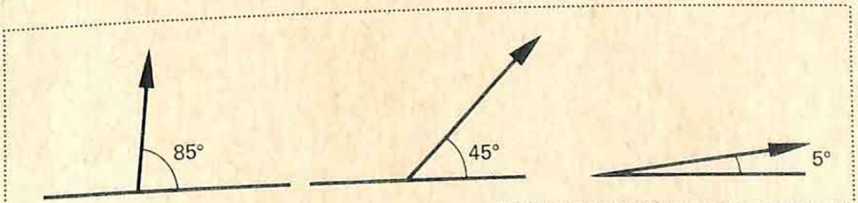
g Describe the path traced out by a stone from the moment it is released from a catapult to the time it reaches the point marked P in the sketch.



h Aristotle thought that this sketch showed the way an arrow flew. Was he right?

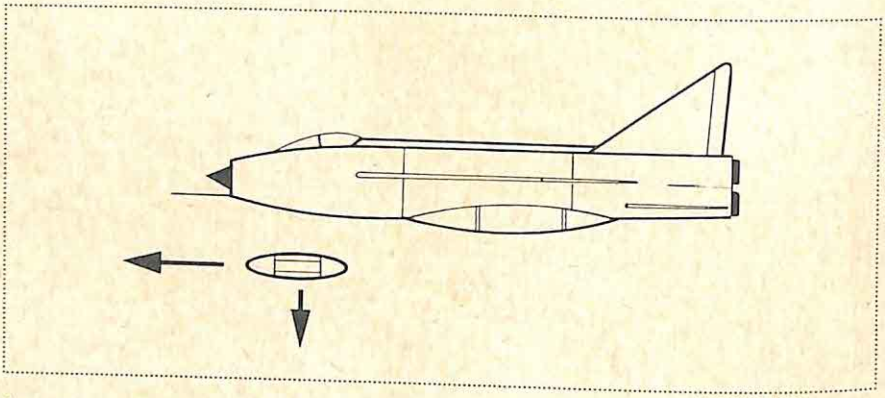


i A boy is experimenting with a tennis ball and he throws it in various ways. Make drawings to show what happens if he throws it in these three ways:



j What do you think the path traced out by a rifle bullet might be?

k A fast moving aircraft drops a package of supplies. The package is moving forward with the aircraft before it is released, so it continues to go forward. It also gains in downward speed all the time it is falling. What sort of path does it trace out as it falls?

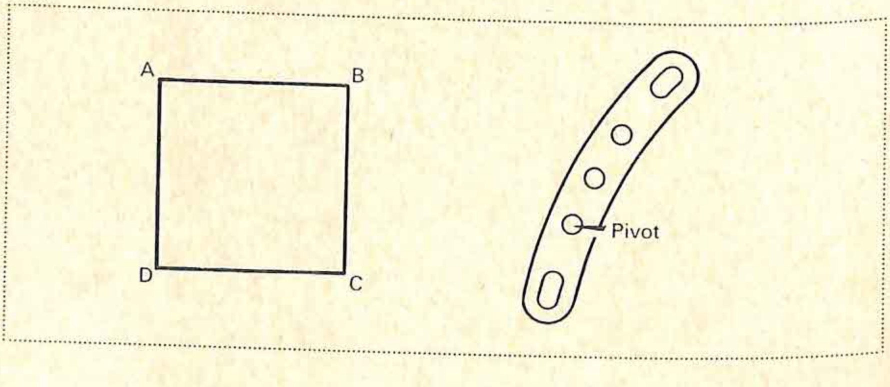


l A golfer swings at a ball. He hits it. Describe approximately the motion of the tip of his iron in the air, and also the path of the ball.

m What is the approximate shape traced out in the air by the valve on a bicycle wheel as the bicycle is ridden forwards on a flat surface? (If you need help, try rolling a penny along and watching the locus of the letter 'O' of 'ONE'; or fit a cotton reel on to a knitting needle and watch the motion of an ink dot on the edge of one face as it is rolled.) What would be the difference if we look at the path traced by a piece of adhesive tape attached half way along a spoke?

n What is the path traced out by the pencil point of a pair of compasses in normal use?

o What is the path traced out by the corner C of a square board ABCD when it is turned about the corner D?

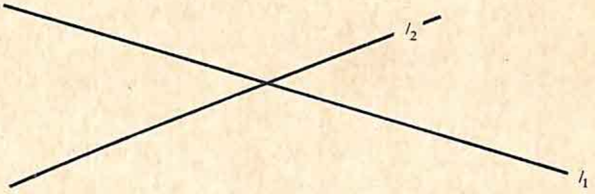


p What is the path traced out by the tips of a curved piece of Meccano, pivoted at a bolt hole?

The paths traced out in each of these examples are really sets of positions of some point or other. A set of positions or a path is sometimes called a *locus*. The plural of *locus* is *loci*.

Exercise 2 Here are some loci for you to draw. Make sure that you find all the positions the point can occupy in each case or you will not have completed the locus.

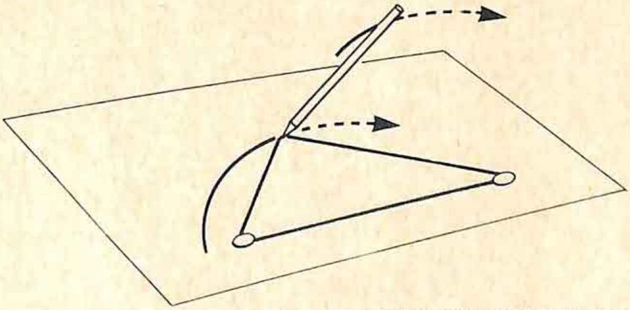
a



Draw two lines intersecting. Now use your protractor and ruler to draw the locus of a point which moves so that it is always the same distance from each of the lines.

b Draw a line segment 8 cm. long. Call it AB. Now use your instruments to find the locus of a point P which is always the same distance from A as B.

c For this you will need a drawing board, two drawing pins, about 60 cm. of fine string or strong thread, a sheet of paper about 30 cms. square and a sharp pencil. Mount the paper on the board near the middle and insert two drawing pins 10 cm. apart. Cut two lengths of string, one 25 cm. and the other 35 cm. and knot them both to make loops. Place the smaller loop round the pins, pull tight with the pencil and draw the locus of the pencil point. Now repeat with the larger loop.



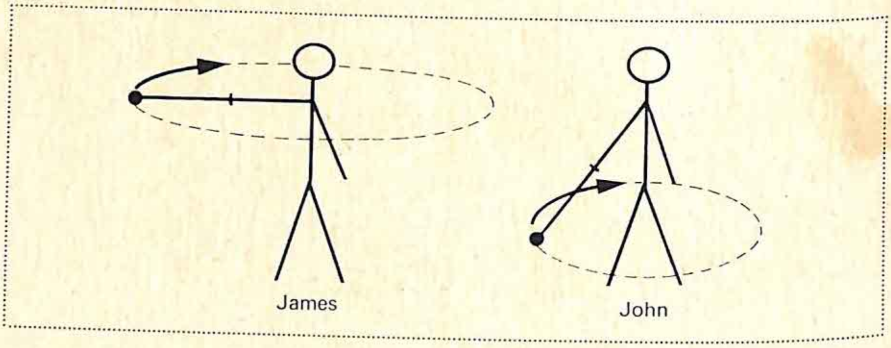
d What is the set of positions of a point (on a flat surface) which is a fixed distance of 1 in. from a ruler edge?

e What is the set of positions on a plane of a point which is a fixed distance of 2 in. from a fixed point?

f What is the set of positions on a plane of a point which is always the same distance from each of two fixed points?

g What is the set of positions on a plane of a point which is always the same distance from each of two fixed straight lines? (Make sure you have all of the picture.) Is there an interesting angle in the diagram?

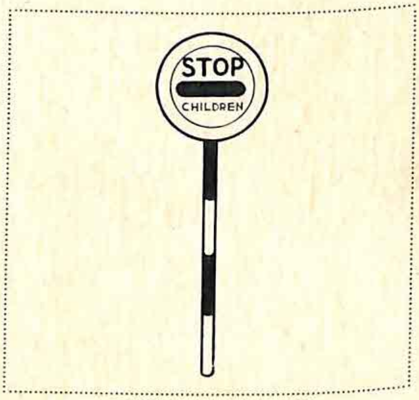
Exercise 3 **a** James and John each have a piece of string on the end of which there is tied a small weight. They each swing the string round.



i What is the locus of the weight in each case?
ii What is the locus of the *string* in each case?

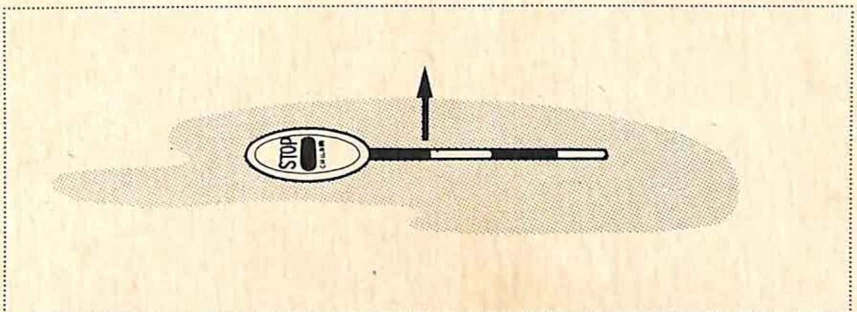
(The string is a segment, a set of points, and we are interested in the set of positions this segment can occupy.)

b Mr Smith is a school crossing patrol. He has a pole with a notice on a circular board.

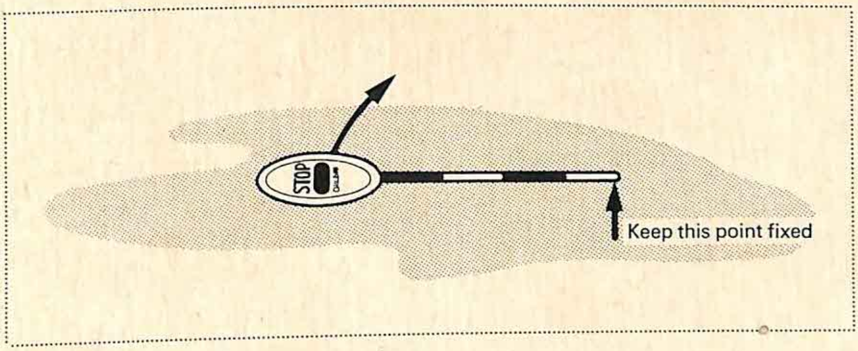


i If he rolls the pole between his hands to rotate the board, what is the shape traced out by the disc?

ii If he were to lay his board down and then lift it as in the sketch, what would the new shape made by the disc be like?



iii If he were to lay the pole down and then move it as in this diagram, what would the shape be like now?

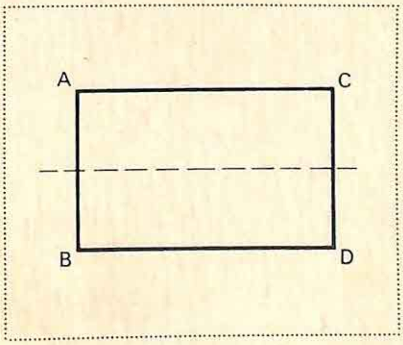


c Turn this page of the book over (when you have read the question!) and say what shape it traces out in the air.

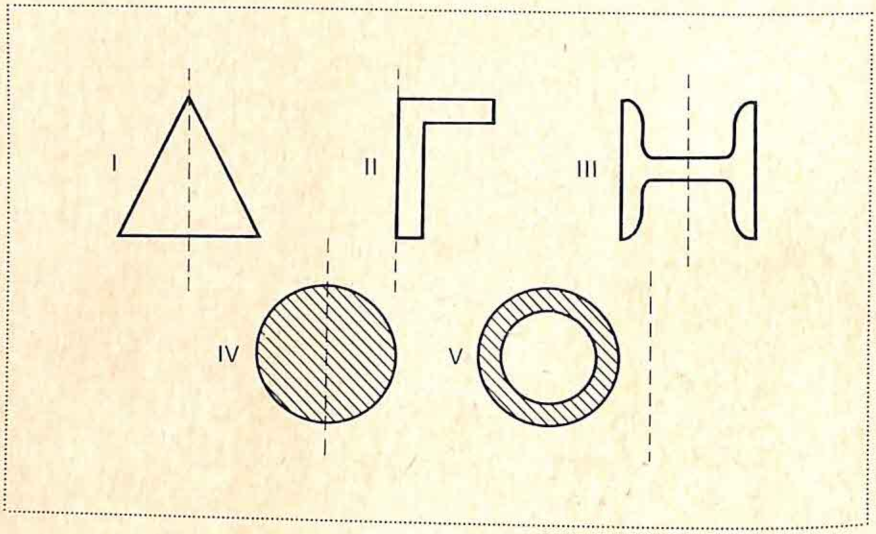
d Here is a rectangle ABCD. What shape does it make when it is rotated

i through 360° about the side AB?

ii Through 180° about the axis dotted in the diagram?



e Sketch the shape made by the movement of each of these shapes if they are lifted in a straight line out of the paper. (Ignore the dotted lines.)



f Sketch the shape made by rotating each of the shapes through 360° about the dotted lines.

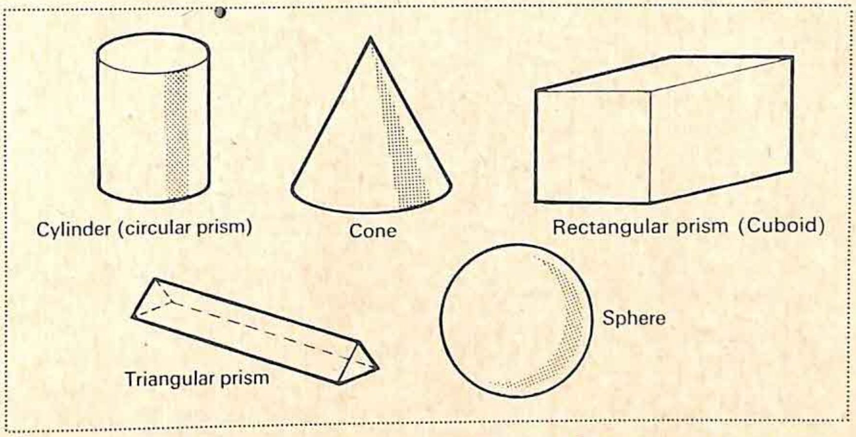
Exercise 4 **a** Draw the shape you would have to lift in a straight line to produce the shape of

i a penny
ii a match box
iii a straight copper pipe
iv a wooden ruler
v a rake handle
vi a railway line

b Draw the shape you would have to rotate to form each of the following shapes. In each sketch show by a dotted line how the rotation is to be done.

i a penny
ii a cone
iii a straight copper pipe
iv a copper pipe bent into a circle and sealed together
v a rake handle
vi a bottle
vii the rim of a bicycle wheel
viii a ball bearing

Here are some of the shapes you have been using. Some of them have special names.



Exercise 5 **a** The locus of all points in space at a distance of 8 in. from a fixed point is – ?

b What is the set of positions in space of a point which is 2 in. from a fixed line?
How long is the locus?

c What is the set of positions of a point which can move on a plane without restriction?

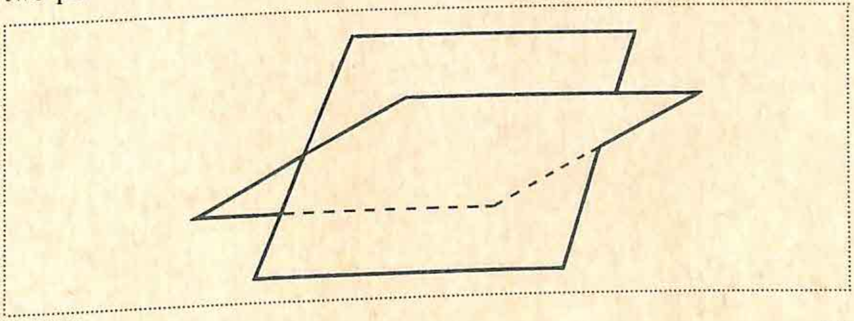
d What is the set of positions of a point which is the same distance from each of three fixed points. Give two answers (i) if the point lies on the same plane as the three points; (ii) if it does not.

e What is the set of positions on a plane of a point which is the same distance from each of four points on the same plane?

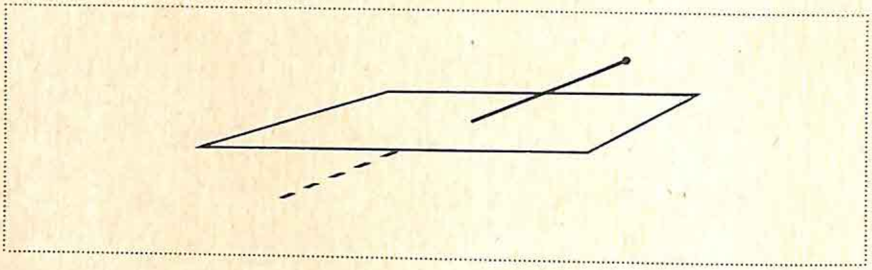
f What is the set of positions on a plane of a point which is the same distance from the four corners of a rectangle on the same plane?

g What is the set of positions of a point which can move anywhere in space?

h What is the locus of a point which can move along the intersection of two planes? Think of the two parts of a folded sheet of paper.



i What is the set of points on the intersection of a line and a plane in space? (Imagine a pin held so that the point is touching your desk.)



j Are there special cases of (h) and (i)?

Section 21 **A special set of fractions**

Exercise 1 Find numbers to fill the frames making true statements of the following:

a $\frac{1}{10}=\frac{\square}{100}$

b $\frac{2}{25}=\frac{\square}{100}$

c $\frac{3}{50}=\frac{\square}{100}$

d $\frac{15}{20}=\frac{\square}{100}$

e $\frac{4}{5}=\frac{\square}{100}$

f $\frac{1}{2}=\frac{\square}{100}$

As we learned before $\frac{3}{8}$ can be expressed as $3\times\frac{1}{8}$. In the same way $\frac{25}{100}$ is $25\times\frac{1}{100}$. So we can see that these fractions, in which the lower number is 100, all mean 'so many hundredths'; $\frac{5}{100}$ means 'five hundredths', for instance. We have a special sign we sometimes use for this: $\frac{5}{100}$ is written as 5%. The % sign is read out as *per cent*.

Exercise 2 Rewrite the following in the alternative form, reducing your fractions to their lowest terms.

a 27%

b 38%

c $\frac{12}{100}$

d $\frac{56}{100}$

e 14%

f 99%

g $\frac{88}{100}$

h $\frac{33}{100}$

i $\frac{100}{100}$

j $\frac{125}{100}$

k 150%

l 175%

You will have noticed that we have not so far been able to change all fractions to percentages. Read this carefully to see how $\frac{1}{3}$ can be changed.

$$\frac{1}{3}=\frac{100}{300}=\frac{100}{3\times 100}=\frac{100}{3}\times\frac{1}{100}=\frac{100}{3}\%=33\tfrac{1}{3}\%$$

Now read this conversion:

$$\frac{3}{7}=\frac{300}{700}=\frac{300}{7}\times\frac{1}{100}=\frac{300}{7}\%=42\tfrac{6}{7}\%$$

Exercise 3 Follow this process to convert the following fractions to percentages.

a $\frac{5}{8}$

b $\frac{6}{7}$

c $\frac{3}{8}$

d $\frac{4}{9}$

e $\frac{7}{8}$

f $\frac{4}{11}$

Exercise 4 You can shorten your writing in these calculations by jumping straight from the first fraction to the percentage:

$$\frac{1}{7} = \frac{100}{7}\% = 14\tfrac{2}{7}\%$$

Use the shortened method in this exercise.
Rewrite the following common fractions as percentages.

a $\frac{3}{10}$
b $\frac{8}{25}$
c $\frac{9}{50}$
d $\frac{x}{100}$
e $\frac{y}{25}$
f $\frac{z}{5}$
g $\frac{4}{3}$
h $\frac{2}{3}$
i $\frac{1}{8}$
j $\frac{9}{8}$
k $\frac{3}{2}$
l $\frac{8}{5}$

Exercise 5 When we say 'of' we mean 'multiply'. Thus 14% of 280 is

$$\frac{14}{100}\times 280 = \frac{392}{10}\% = 39\tfrac{1}{5}\%$$

Work out

a 15% of 30
b 27% of 450
c 96% of 200
d 98% of 370
e 125% of 50
f 200% of 980

If we want to give 35 as a percentage of 140, we write it first as a fraction, $\frac{35}{140}$, and then convert the fraction to a percentage in the usual way:

$$\frac{35}{140} = \frac{350}{14}\% = 25\%$$

Exercise 6 **a** Write the first number as a percentage of the second in each of the following.

i 15; 90
ii 26; 130
iii 58; 754
iv 12; 144
v 130; 3900
vi 13; 78

b Now write the second number as a percentage of the first in the same set of number pairs.

We can take percentages of quantities of things. It is very common to take percentages of sums of money, or of quantities of merchandise. The process is exactly the same as with numbers.

$$5\% \text{ of } £200 = £\frac{5}{100}\times 200 = £10$$

Exercise 7 Find the following:

a 5% of £300

b 12% of £700

c 76% of £600

d 65% of £650

e 27% of 400 tons

f 84% of 1000 lb.

Section 22 **Powers**

Exercise 1 **a** What do the following mean?

i ab ii $a \, . \, b$ iii $a \times b$

b Explain the meanings of

i 23 ii 2 . 3 iii 2×3 iv 2·3

c Compare (a)i with (b)i. What do you see?

d Can you write any of the expressions in (a) and (b) another way? Do so.

e Shorten

i $2 \times c$
ii $2 \times 3 \times c$
iii $2 \times c \times d$
iv $2 \times 2 \times c$
v $4 \times 5 \times t$
vi $3 \times u \times v$
vii $9 \times 8 \times i \times j \times k$
viii $45 \times 3 \times x \times y \times z$

f Explain exactly what each of the following mean.

i $2cd$
ii $7efg$
iii $10xyz$
iv $24pq$
v $4 \times (5 \times 6)$
vi $7e(f \times g)$
vii $10z(x \times y)$
viii $(10y \times x) \times z$

g Can any of (f) be shortened?

We can write multiplication without the multiplication signs when only one known number is involved. If we have more than one number, we must multiply them first or put in the sign.

Exercise 2 **a** How many times can we subtract

i 5 from 15?
ii 7 from 56?
iii 12 from 144?
iv 9 from 7×9?
v 5 from 24×5?
vi 6 from 6×17?
vii 8 from 18×8?
viii c from $3 \times c$?
ix d from $7d$?
x $2d$ from $2d \times 6$?
xi $3g$ from $8 \times 3g$?
xii $3g$ from $33g$?
xiii $5t$ from $50t$?
xiv $12x$ from $96x$?
xv $60y$ from $600y$?
xvi $100z$ from $1000z$?

b i What is $4c-4c$?
ii Find $4c-2c$.
iii What is half of $4c$?
iv $4c \div 2 = ?$
v $\dfrac{2\times 2c}{2} = ?$
vi $\dfrac{4c}{2} = ?$
vii $\dfrac{4}{2}\times c = ?$
viii $4\times\dfrac{c}{2} = ?$

c i How many times can you take c from $3c$?
ii $(c+c+c)-c-c-c = ?$
iii Find $\dfrac{3\times c}{c}$
iv What is $3c \div c$?

d Find the following:
i $\dfrac{4\times c}{c}$
ii $4c\div c$
iii $\dfrac{17\times d}{d}$
iv $\dfrac{8g}{g}$
v $\dfrac{18f}{f}$
vi $\dfrac{18f}{2}$
vii $\dfrac{18f}{2f}$
viii $\dfrac{18f}{18f}$

e Simplify the following:
i $4\times 5\div 5$
ii $\dfrac{4\times 5}{5}$
iii $7\times 6\div 7$
iv $7\times 6\div 6$
v $3\times 8\div 3$
vi $9\times 5\div 5$
vii $18\times 5\div 9$
viii $27\times 8\div 24$
ix $81\times 7\div 63$
x $100\times 7\div 35$
xi $56\times 4\div 32$
xii $49\times 8\div 56$
xiii $6d\div d$
xiv $12f\div f$
xv $12f\div 2f$
xvi $24g\div 8g$
xvii $32h\div 4h$
xviii $78j\div 4j$
xix $51k\div 17k$
xx $95m\div 19m$
xxi $18mn\div 9m$
xxii $24jk\div 8k$
xxiii $34xy\div 17xy$
xxiv $136xyz\div 17xz$

f Shorten
i $\dfrac{6d}{d}$
ii $\dfrac{6d}{2d}$
iii $\dfrac{6\times 7}{2\times 7}$
iv $\dfrac{7e}{2e}$
v $\dfrac{7\times 5}{3\times 5}$

g Find the values of the following:
i $(7\times 5)\div(3\times 5)$
ii $(7\div 3)\times 5$
iii $7\div(3\times 5)$
iv $7\times(5\div 3)\div 5$
v $7\times(5\div 3)\times 5$
vi $(7\div 5)\times(3\div 5)$
vii $(7\div 5)\times(5\div 3)$
viii $7\div(5\div 3)\times 5$

h Simplify

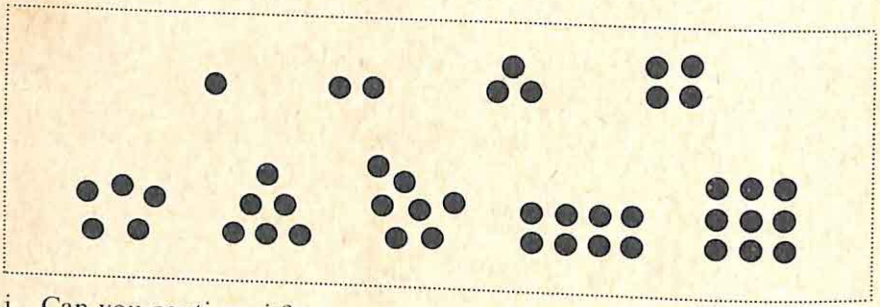
i $a\times(b\div c)\times b$
ii $a\times(b\div c)\div b$
iii $(a\div c)\times b$
iv $a\div(b\times c)$
v $a\div(b\div c)\times b$
vi $a\div(b\times c)\div b$
vii $a\times(b\times c)\div b$
viii $a\div(b\times c)\times b$

Exercise 3 **a** i What do you notice about the following two sequences?

Sequence A: 1, 2, 3, 4, 5, 6, . . .
Sequence B: 1, 4, 9, 16, 25, . . .

ii What is the next number in B?
iii What are the next three numbers in each sequence?
iv What is the area of a square whose side is 4 in.?
v What is the area of a square whose side is 6 cm.?
vi What have the last two questions to do with the sequences?

b Look at this sequence:



i Can you continue it?
ii Which sets of dots can be arranged in triangles?
iii Which sets of dots can be placed in square formations?
iv What have the following three sequences to do with the dots?

Sequence C: 1, 3, 6, 10, . . .
Sequence D: 1, 4, 9, . . .
Sequence E: 2, 5, 7, 8, . . .

v Extend each of these sequences by at least two numbers.
vi If you were asked what triangular numbers were, what would you say?
vii What are square numbers?
viii How are the square numbers connected with the natural numbers?

c Could you make up a sequence of rectangular numbers?

d i Write out sequence D for 7 terms and find the first differences:

Sequence D: 1, 4, 9, . . .
$\triangle^1$: 3, 5, . . .

ii Write out the sequence of differences beyond the point where D finishes. Can you use these differences to build up the values which should be in D?

e If we say the number 4 is 2 *squared*, what will we call 16?

f Write down the values of

i 5 squared   ii 3 squared   iii 7 squared

g Rewrite the first six terms of sequence D as *something squared*.

h We write down 4 squared as 4^2. Rewrite the first six terms of D in this form.

i What are the values in ordinary notation of

i 9^2	iii 13^2	v 17^2	vii 23^2
ii 10^2	iv 15^2	vi 21^2	viii 30^2?

j Write down a squared and x squared in this notation.

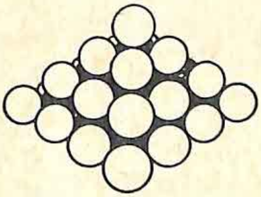
k How could you write down (i) $y \times y$ and (ii) $z \,.\, z$?

l Can you guess a possible meaning on the same lines for 4^3 and 2^5?

m Can you rewrite the following in a shorter notation?

i $5 . 5 . 5 . 5$	iv $y . y . y . y$
ii $2 \times 2 \times 2 \times 2 \times 2 \times 2$	v $z . z . z . z . z . z$
iii $x . x . x$	vi $a . a . a . a . a . a . a . a$

Exercise 4 The sketch shows how you might pile up balls or spheres into the shape we call a pyramid. In fact this was the way that cannon balls used to be stored.

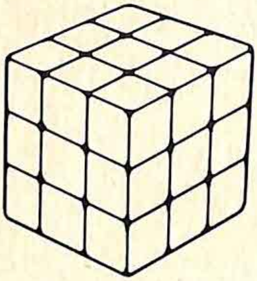


a Write down a sequence of pyramidal numbers. (The first is 1 and the second is 5; the bases are square.)

b Do you think you could build pyramids using equilateral triangles as the base? If so, write out the sequence you obtain. Clearly the first number will again be 1.

c What is the connexion between the last two sequences and the sequences of square and triangular numbers?

Exercise 5 **a** This sketch shows a set of sugar lumps arranged in the shape of a cube.



i Which different numbers of sugar lumps could be arranged to form cubes?

ii Write out the sequence of cubic numbers.

iii Can you use the kind of abbreviation you used in Exercise 3(1) to express the numbers of this sequence?

Exercise 6 **a** The sequence 2, 2^2, 2^3, 2^4, 2^5, etc. is read out as *two, two squared, two cubed, two to the fourth power, two to the fifth power* and so on. (Most people leave out the word 'power' however.)
Work out this sequence in ordinary notation up to 2^{10}.

b Work out this sequence up to the sixth term.
3, 3^2, 3^3, 3^4, 3^5, ...

c The last two sequences are sometimes called *power* sequences. Work out the first six terms of the power sequence of 5.

d i Work out $2^4 \times 2^3$

ii Find $2^3 \times 2^5$

iii What is $2^5 \times 2^5$?

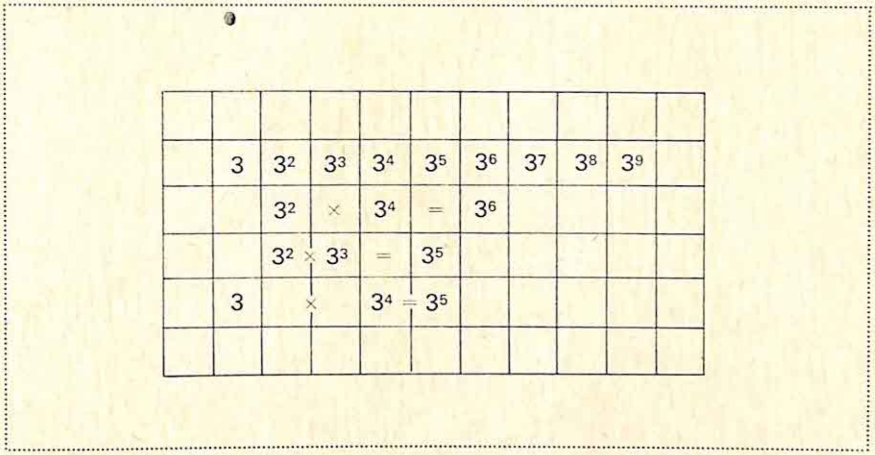
iv Calculate $2^4 \times 2^4$.

e Do the answers to (d) appear in the set of answers to (a)?

f Do the answers to these appear in the answers to (b)?

i $3^4 \times 3^2$ ii $3^2 \times 3^3$ iii 3×3^4

Exercise 7 Look at this array of numbers. The top row is a power sequence of 3. Below we have written some of the calculations you have just done.



a Copy the diagram onto squared paper. Now put in the following questions together with their answers:

i $3^3 \times 3^5$
ii 3×3^2
iii $3^2 \times 3^6$
iv $3^3 \times 3^6$
v 3×3^8
vi $3^7 \times 3^2$
vii $3^6 \times 3$
viii $3^5 \times 3^3$

b Which law of natural numbers have we had to assume in putting some of these on to the diagram (e.g. vi)?

c i The power sequence is a set. Is it finite or infinite?
ii The sequence is a set which is *closed for multiplication*. What do you think this means?
iii Is the set closed for division?
iv Is it closed for addition?
v Is it closed for subtraction?

d Can you see a rule for multiplying pairs of numbers like those in (a) above?

e Can you find a corresponding rule for division?

f Is there a similar rule for (i) addition or (ii) subtraction?

g Use your rules for working these questions out and then check by ordinary arithmetic.

i $3^5 \times 3$
ii $3^2 \times 3^4$
iii $3^4 \div 3^2$
iv $3^5 \div 3^3$

h i Could you do a similar kind of thing for a sequence of powers of, say, 2 or 7?
ii Would your rules for multiplication and division be similar or different?

Exercise 8 **a** Work out the following leaving the answers in power form. You may check the answers to alternate questions at the end of this section.

i $2^4 \times 2^6$
ii $3^5 \times 3^7$
iii $5^5 \times 5^6$
iv $3^9 \times 3^8$
v $2^7 \times 2^6$
vi $7^9 \times 7^9$
vii $4^8 \div 4^6$
viii $3^9 \div 3^8$
ix $2^7 \div 2^5$
x $5^9 \div 5^4$
xi $6^{12} \div 6^3$
xii $9^9 \div 9^5$
xiii $8^7 \times 8^7$
xiv $7^3 \times 7^5$
xv $6^5 \div 6^3$
xvi $1^{12} \times 1^{14}$
xvii $12^{14} \div 12^{12}$
xviii $11^{11} \div 11^1$
xix $4^8 \div 4^4$
xx $9^9 \div 9^3$
xxi $3^9 \div 3^3$
xxii $4^5 \times 4^7$
xxiii $7^7 \div 7^4$
xxiv $5^4 \times 5^2$

b Read and check each of the following statements:

i $2^4 \times 2^3 = 2^7$
ii $(2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$
iii $2^5 \div 2^3 = 2^2$
iv $\dfrac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2 \times 2 = 2^2$
v $2^{(\text{any number})} \times 2^{(\text{any number})} = 2^{(\text{sum of the numbers})}$
vi $2^{(\text{any number})} \div 2^{(\text{any smaller number})} = 2^{(\text{difference of the numbers})}$
vii $2^x \times 2^y = 2^{x+y}$
viii $2^x \div 2^y = 2^{x-y}$, when $x > y$
ix $a^x \times a^y = a^{x+y}$
x $a^x \div a^y = a^{x-y}$, when $x > y$

c Shorten the following in power notation:

i $5^4 \times 5^3$
ii $7^4 \times 7^{10}$
iii $6^{14} \times 6^{20}$
iv $10^4 \times 10^3$

d Use power notation to work out these, leaving the answer in power form.

i 8×16
ii 27×81
iii 16×64
iv 5×125

e Some of the following are wrong. Explain what is wrong with them:

i $3^2 \times 3^3 = 3^6$
ii $4^5 \times 4^4 = 4^9$
iii $3^5 \times 4^5 = 12^5$
iv $3^5 \times 4^5 = 7^5$
v $5^4 \times 5^3 = 5^7$
vi $4^5 \times 4^3 = 4^2$

f Simplify each of these in power notation and leave your answers in power form.

i $5^5 \div 5^2$
ii $125 \div 25$
iii $128 \div 16$
iv $81 \div 27$
v $10000 \div 100$
vi $1{,}000{,}000{,}000 \div 100{,}000$

g Simplify if you can. In iv and v, say what must be true for your rules to work.

i $3^5 \div 5^3$
ii $a^6 \div a^2$
iii $a^3 \div a^6$
iv $a^x \div a^y$
v $a^z \div b^y$
vi $3^8 \div 3^7$

Exercise 9 **a** i What is the volume of a rectangular block *a* units long, *b* units high, and *c* units wide?
ii What is the volume of such a block *a* units each way?
iii What is the name of such a shape?

b Write down in power form the volumes of the following blocks:

i 7 in. each way
ii 10 cm. each way
iii 13 cm. each way
iv 2 in. by 2 in. by 4 in.
v 3 cm. by 9 cm. by 27 cm.
vi 5 cm. by 125 cm. by 25 cm.

c These figures are for the volumes of cubes. How long is the side in each example?

i 125 cu. in.
ii 27 c.c.
iii 64 c.c.
iv 343 c.c.

d If $x^2 = 81$, what is x? 81 is said to be the *square* of the number x, while x is said to be the *square root* of 81.

e If $x^3 = 64$, what is x? 64 is said to be the *cube* of the number x, while x is said to be the *cube root* of 64.

f Find the square roots of each of the following.

i 16
ii 36
iii 25
iv 1
v 0
vi 100
vii x^2
viii y^8

g Find the cube roots of the following.

i 8
ii 125
iii 27
iv 216
v x^3
vi y^{12}

h See if you can simplify all of the following, leaving the answers in power notation.

i $9^3 \times 9^3$
ii $(9^3)^2$
iii $2^2 \times 4$
iv $2^2 \times 2^2$
v $(2^2)^2$
vi 2×4^2
vii $3^2 \times 3^2 \times 3^2$
viii $(3^2)^3$
ix $4^2 \times 4^3 \times 4^4$
x $5^4 \times 5^3 \div 5^5$
xi $6^4 \div 6^5 \times 6^3$
xii 216×1296

Answers to Exercise 8 **a**

i 2^{10}

iii 5^{11}

v 2^{13}

vii 4^2

ix 2^2

xi 6^9

xiii 8^{14}

xv 6^2

xvii 12^2

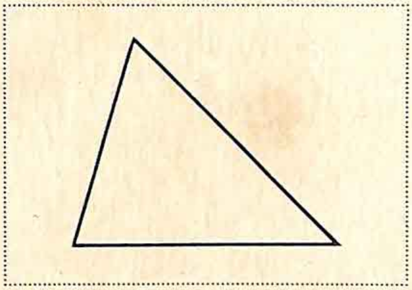
xix 4^4

xxi 3^6

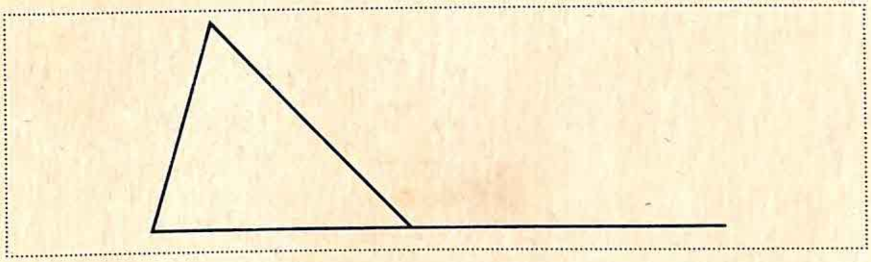
xxiii 7^3

Section 23 **Congruent triangles**

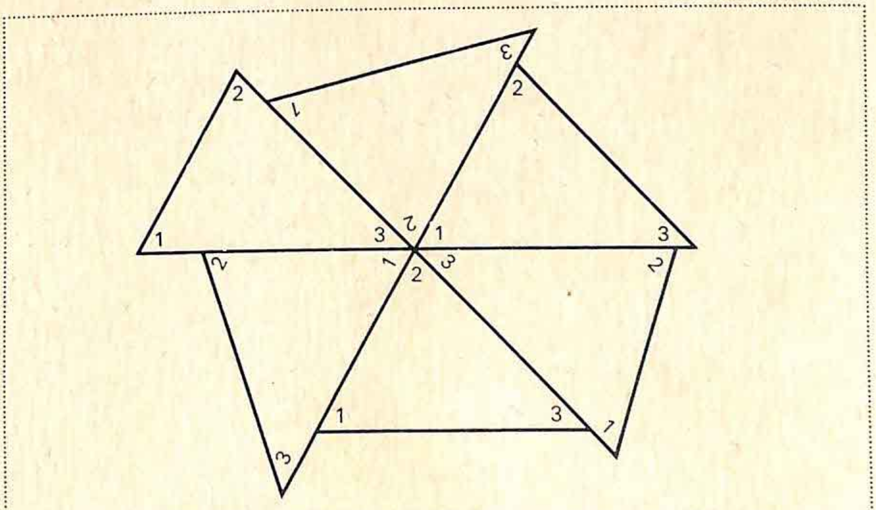
For the work of this section you will need to cut out a set of six congruent triangles from stout paper. The size and shape of the triangles does not really matter but you will find that the following are convenient dimensions: 3 in., $3\frac{1}{2}$ in., 4 in.



Exercise 1 **a** Take three of your triangles. In how many ways can you arrange them to form a straight line?

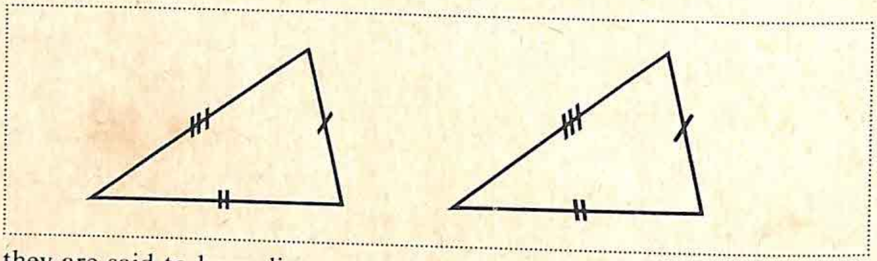


b Use all six triangles.



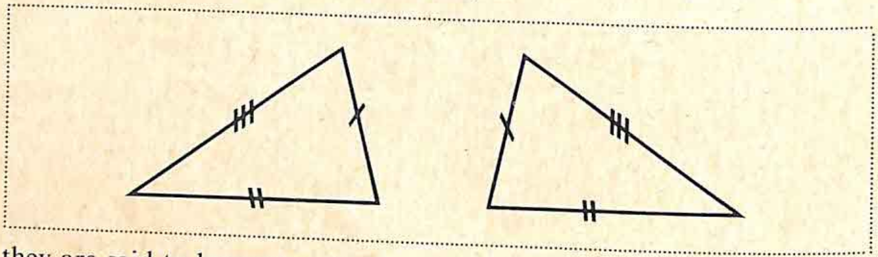
i In how many ways can you group them round a point?
ii Do you always get straight lines across the figure?
Explain.

When triangles are congruent like this:



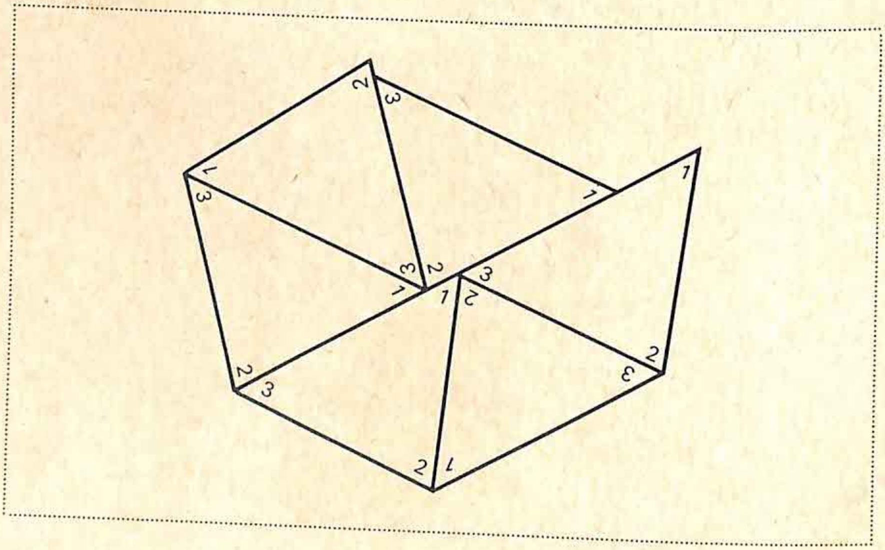
they are said to have direct congruence.

When triangles are congruent like this:



they are said to have opposite congruence.

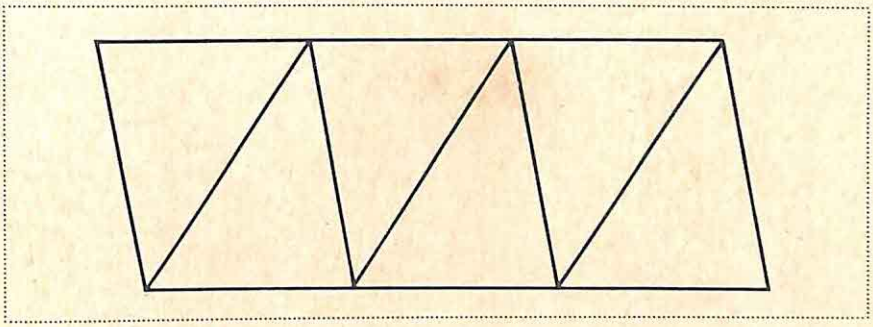
Exercise 2 Arrange your six triangles to look like this.



a How many pairs of parallel lines can you find in the diagram?

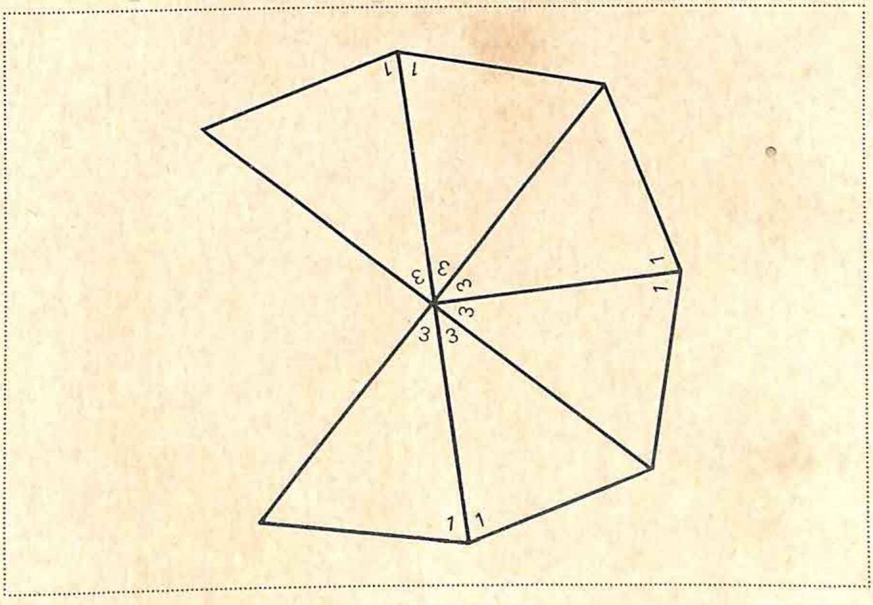
b How many of these triangles are directly congruent?

Exercise 3 **a** Arrange your triangles like this.



i How many *pairs* of triangles have direct congruence?
ii How many pairs of triangles have opposite congruence?
iii How many sets of parallel lines are there and how many parallels are there in each set?

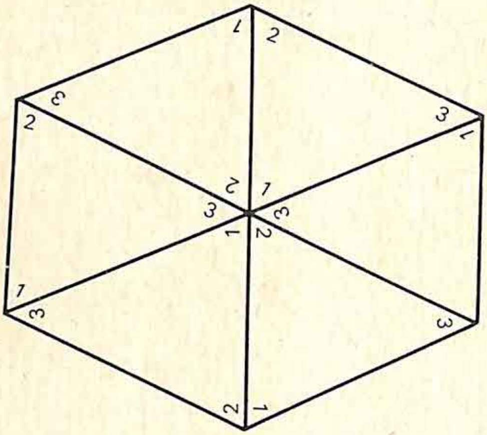
b Arrange your triangles like this.



i How many pairs of triangles have opposite congruence?
ii How many pairs of triangles have direct congruence?

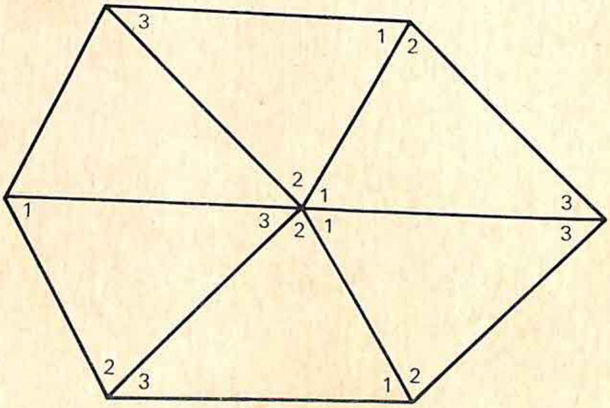
Exercise 4 **a** How many ways can you arrange your six triangles to form hexagons?

b Here is one hexagon.



i How many parallelograms are there in the figure?
ii How many straight lines are there in the figure?
iii How many pairs of vertically opposite angles can you find?
iv How many triangles are directly congruent?
v Could you cover a plane entirely with a set of hexagons like this one?

c Here is another hexagon.



i How does this hexagon differ from the previous one?
ii How many parallelograms are there in the figure?
iii How many straight diagonals are there in the figure?

iv How many pairs of vertically opposite angles are there in the figure?
v How many pairs of alternate angles are there in the figure?
vi Partition the set of triangles into two sets according to their congruence. How many are there in each set?
vii Could you cover a plane entirely with a set of hexagons like this one? Explain.

Section 24 **Games with numbers**

Exercise 1 This is a table of multiplications to the base 10. Draw up your own copy. Do it on $\frac{1}{4}$ in. or metric squared paper. Put the numbers where the lines cross.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300
16	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320
17	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340
18	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360
19	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380
20	20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400

a Choose a number which appears more than twice in the table. Ring the points at which this number appears. Now with a sharp coloured pencil draw a smooth line through the points you have ringed. The line you have drawn is a curve called a *rectangular hyperbola*. What sort of curve do you get by selecting another number and doing the same thing again?

b Now draw the diagonal of your square from the number 1.

i What are the numbers on this line?

ii Write the first five numbers out in a different notation.

iii Write out the sequence and the first and second differences as we have begun to do here. Continue to the end of the sequence.

sequence: 1 4 9 16 25 36 49 64 …
$\triangle^1$: 3 5 7 9 11 …
$\triangle^2$: 2 2 2 2 …

iv Can you use this to extend the table?

v Use this table to find the values of
$17^2, 16^2, 14^2, 13^2, 19^2, 23^2, 25^2, 27^2$.
(Check the last two by multiplication.)

c Now try taking any line of numbers parallel to the diagonal.
We have chosen 4, 10, 18, 28, 40 …
$\triangle^1$: 6 8 10 12 …
$\triangle^2$: 2 2 2 …

Can you use this to predict new numbers? Try it with several sequences parallel to the diagonal.

d Now try this. Start from any number in the table. Write it down. Now count one place to the right and two down, and write down the number. Now one place to the right and two down again and so on.
We have chosen 4 in the top edge.

Sequence: 4 15 30 49 …
$\triangle^1$: 11 15 19 …
$\triangle^2$: 4 4 …

Find the differences and continue as before. Can you begin to predict numbers instead of reading them from the table?

e Try to find more patterns in the numbers by similar means, e.g. three across and one up or one down.

f If the instructions are 'two to the right, three down' would it make any difference if you said 'three down, two to the right'? Can you always reverse operations in this way? Can you reverse

i $2+8$ ii 5×6 iii $8-5$ iv $12\div 6$

v opening a book and reading it
vi putting on a sock then a shoe
vii cleaning your teeth and dressing

viii cooking your lunch then eating it,
without altering the result?

g Examine the line labelled first differences, $\triangle^1$, in each example you have worked out. Each one is what is called an *arithmetic progression.* What is the important thing about arithmetic progressions you can see by looking at the second differences, $\triangle^2$.

h What patterns can you get out of the diagonal from bottom left to top right and the lines parallel to it?

Exercise 2 **a** Look back at the table which began this section. The column third from the left is 3, 6, 9, 12, etc. This is a set of numbers. Does it fit the description $\{x \mid x \text{ is a multiple of } 3;\ 3 \leqslant x;\ x \text{ is a natural number}\}$?

b Which set from the table am I describing if I write down $\{x \mid x \text{ is a multiple of } 6;\ 6 \leqslant x\}$?

c Write down the set $\{y \mid y \text{ is a multiple of } 7;\ 14 \leqslant y \leqslant 70\}$.

d Describe the set $\{7, 14, 21, 28, 35\}$.

e Describe the set $\{7, 14, 21, 28, 35, \dots\}$.

Exercise 3 **a** Write down the set $\{p \mid p \text{ is a multiple of } 5;\ 0 < p\}$ putting in fourteen terms and completing with dots. Call it P.

b Write down the set $\{q \mid q \text{ is a multiple of } 6;\ 0 < q\}$ putting in fourteen terms again. Call it Q.

c i Are there any members common to each of the last two sets?
ii Would there be any more members common if you were to write out more of the sets?
iii Write out a set of common members of the two sets. Call it R.
iv What do we call the set which is the common part of two other sets?
v Which set is $P \cap Q$?

d P was a set of multiples; Q was also a set of multiples. Can you now give a definition of the set R in terms of 5 and 6? The set R is said to be the set of common multiples.

e i Do you think that the sets P, Q, R are *ordered* sets?
ii Is there a least member of each of these sets?
iii What is the least common-multiple in this example?
iv What does *least common-multiple* actually mean?

f Write down the following.

i A = $\{a \mid a \text{ is a multiple of } 4; 0 < a\}$
ii B = $\{b \mid b \text{ is a multiple of } 9; 0 < b\}$
iii $A \cap B$
iv the least common-multiple (LCM) of 4 and 9.

g By the same means find the LCM of 18 and 27.

Exercise 4 **a** A divisor of 12 is a number which will divide into 12. Write down the set of all the whole divisors of 12 and the set of all the whole divisors of 16. Call the first set A and the second set B.

b i Write down the set $A \cap B$. This is the set of common divisors of 12 and 16.
ii Is each of the sets A, B, $A \cap B$ ordered?
iii What are the least and greatest members of $A \cap B$?
iv ‘Every number divides by .’ Complete.
v ‘The greatest common divisor (GCD) of 12 and 16 is .’ Complete.
vi How is the GCD connected with 12 and 16?

You may check the answers to some of (c) and (d) at the end of the section.

c By any means you wish find the LCM of

i 3 and 5
ii 2 and 7
iii 22 and 26
iv 15 and 24
v 28 and 52

d Use any method you wish to find the GCD of

i 20 and 24
ii 90 and 120
iii 105 and 135
iv 154 and 455
v 525 and 825

Exercise 5 Common multiples are used in adding and subtracting common fractions.

a i Complete this example.

$\frac{1}{6}+\frac{3}{8}=\frac{}{48}=$

ii What did you notice at the last step?
iii Could you have shortened this by using the LCM of 6 and 8? If so, do it again by the shorter way.

b Complete these examples and then find a shorter way.

i $\frac{11}{15}-\frac{1}{9}=\frac{}{135}=$
ii $\frac{4}{8}-\frac{3}{9}=\frac{}{72}=$

c Work these out by the shortest way.

i $\frac{2}{5}+\frac{5}{15}$
ii $\frac{2}{7}+\frac{6}{21}$
iii $\frac{7}{8}-\frac{5}{6}$
iv $\frac{3}{4}+\frac{5}{2}$
v $\frac{1}{4}-\frac{3}{8}+\frac{7}{12}$
vi $\frac{5}{16}+\frac{4}{12}-\frac{5}{8}$

Answers to Exercise 4

c i $A = \{3, 6, 9, 12, 15, \ldots\}$
$B = \{5, 10, 15, 20, \ldots\}$
$A \cap B = \{15, \ldots\}$
LCM is 15
ii LCM is 14

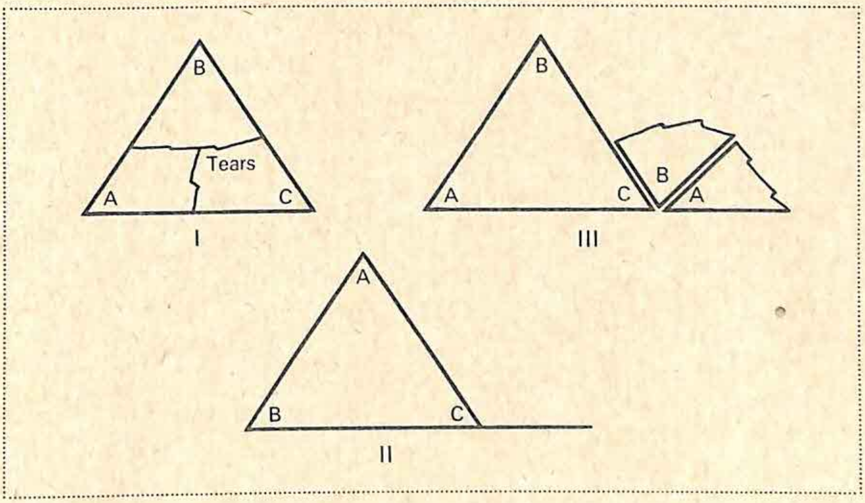
d i $A = \{1, 2, 4, 5, 10\}$
$B = \{1, 2, 3, 4, 6, 12\}$
$A \cap B = \{1, 2, 4\}$
GCD is 4
ii GCD is 30

Section 25 **Geometry with paper and scissors**

For this section you will need some coloured adhesive paper, a pair of scissors, your instruments and your exercise book. When you want to name a figure you will find the list at the end of the section useful.

Exercise 1 **a** Draw a triangle ABC on a piece of adhesive paper. Label the vertices *inside* the triangle (figure I). With the point of your compasses prick through the three vertices into your exercise book and then draw the triangle. Produce any one side of it (figure II). Now tear the adhesive triangle as shown in figure I, and try to place two of the pieces as in figure III.

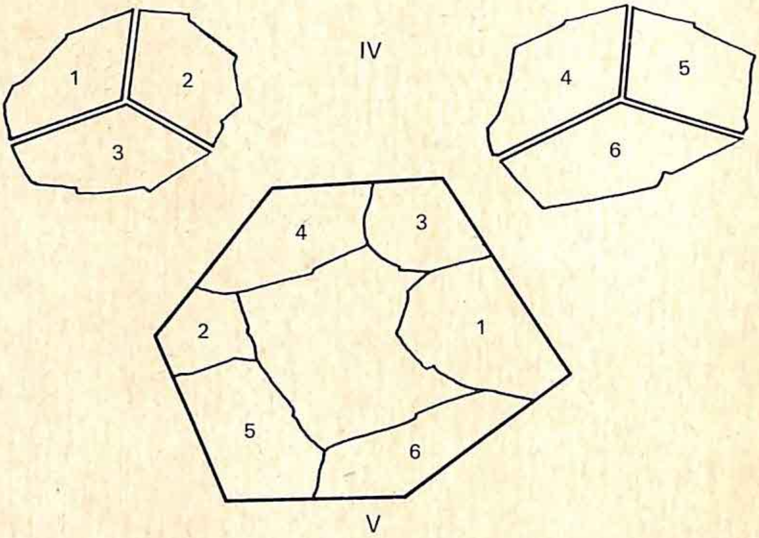
b Repeat with another triangle and a different vertex.



c If you produce two sides at a vertex, what can you say about the angles that are formed?

Exercise 2 **a** Tear two irregular 'rounds' of adhesive paper. Cut each into three pieces as in figure IV. Make sure that you have three obtuse angles. Number the pieces. Can you now arrange the pieces as in figure V (allowing overlaps)? Does the order of the pieces matter? When you have found a solution, stick the result into your book.

b What happens if you have acute angles in the cutting up of the rounds?

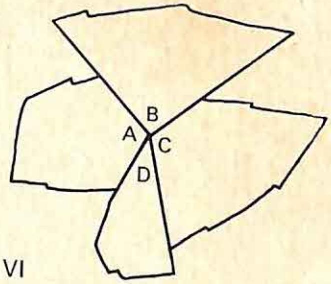


c i What is the sum of the angles in each of the rounds in degrees and in right angles?

ii What is the sum of the angles of the figure you have formed?

iii What is the name of the figure you formed?

Exercise 3 On adhesive paper, draw a quadrilateral and cut it out. Can you now tear off the corners and fit them together as in figure VI? When you have done so, stick the result into your book.



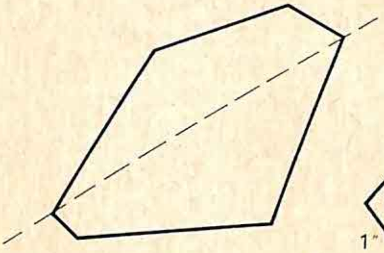
VI

Exercise 4 **a** Draw a hexagon as in figure VII. You may make it any shape or size. Cut it out and then cut it into two quadrilaterals as we have shown. Tear it up and rearrange the pieces to form two rounds like figure IV. Is there more than one way of doing it?

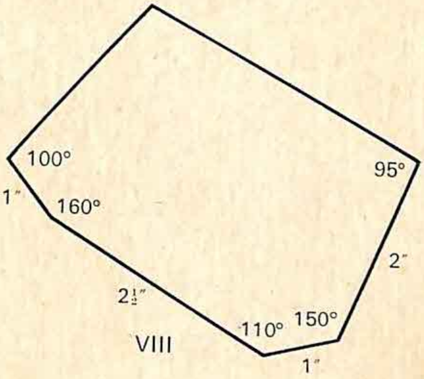
b The angles inside a figure are called *interior* angles. What is the sum of the interior angles of a hexagon?

c i Do you think that you could draw a hexagon, rip off the corners as in figure V and rearrange the pieces as in figure IV?

ii Try with this hexagon. Draw it accurately. Explain your result.



VII



VIII

iii Can you find more hexagons which behave like figure VIII?

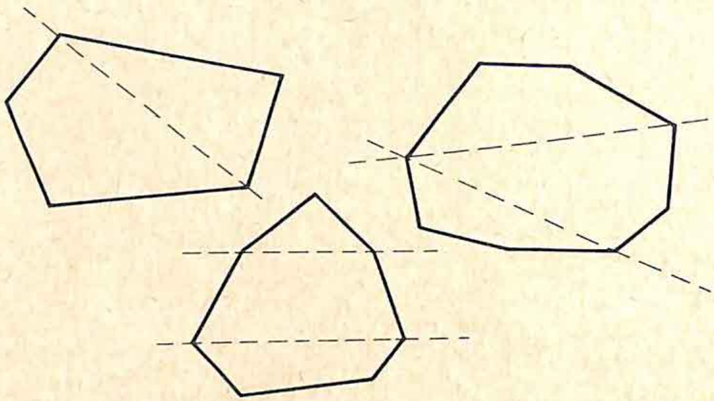
Exercise 5 Figure IX shows three polygons.

a What are the names of these polygons?

b The dashed lines show ways in which you could cut the figure up to make quadrilaterals and triangles. Can you without actually cutting them up, work out the sum of the interior angles of each one? Do so.

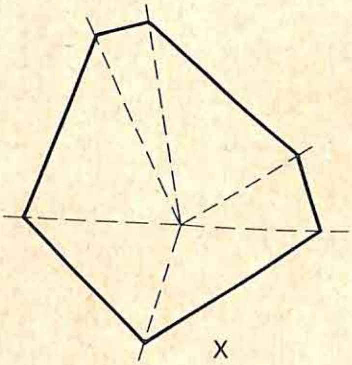
c Could you still do it if you only knew the sum of the angles in a triangle? Draw a sketch to illustrate your answer.

d By any method you please find the sum of the interior angles of a nine-sided polygon and a ten-sided polygon.



IX

Exercise 6 **a** Draw a polygon of more than four sides on your adhesive paper. Cut it out. Choose any point inside it and join the vertices to that point as has been done in figure X. Now mark the angles at the point. Cut out all the triangles.

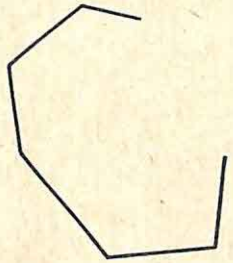


X

i How many sides did your polygon have?
ii How many triangles have you?
iii What is the sum of the angles in a triangle?
iv What is the sum of all the angles in this set of triangles?
v What is the sum of the angles you have marked?
vi What is the sum of the other angles?
vii What was the sum of the angles in your original polygon?

b Make a list of the sums of the interior angles of all the polygons you have met in this section and beside each the number of sides and the sum of the interior angles (in right angles). Can you see a rule for finding the sum when you know the number of sides?

c Here is a figure (figure XI). It is a polygon but we do not know how many sides it has so we cannot complete the drawing.



XI

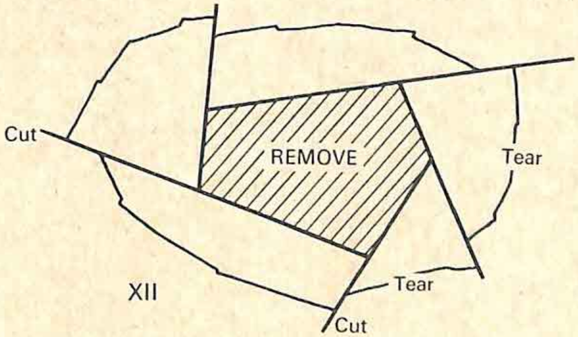
We will say it has n sides. Now answer these questions giving the angles in right angles.

i How many triangles will there be if we cut it up as in (b)?
ii What will be the sum of all the angles in this set of triangles?
iii What is the sum of the angles at the point inside the triangle?
iv What is the sum of the rest of the angles?

d Use the rule you have just found to find the sum of the interior angles of

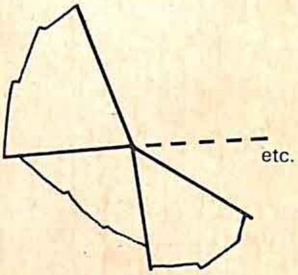
i a quadrilateral
ii an octagon
iii a decagon
iv a 12-gon
v a 20-gon
vi a 72-gon

Exercise 7 Figure XII shows a pentagon. Cuts have been made along the straight lines and tears along the jagged ones.



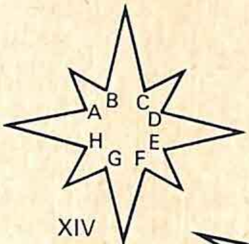
XII

a On a piece of adhesive paper draw a polygon of any number of sides. Produce the sides in order (as has been done on the figure) and cut along the straight lines. Tear the pieces out as the figure shows. Rearrange them as in figure XIII and stick them into your book. What do you notice?

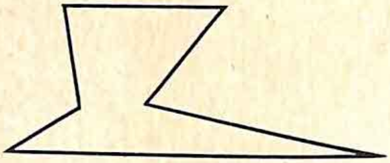


XIII

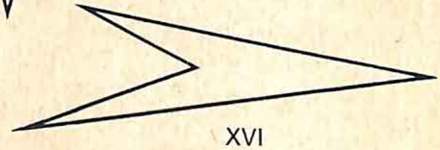
b The angles you cut out are said to be the exterior angles of the polygon. Notice that the sides were *produced in order*. What is the sum of the exterior angles of your polygon? Find out if this is true of other polygons.



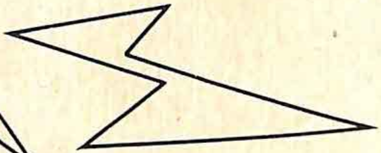
XIV



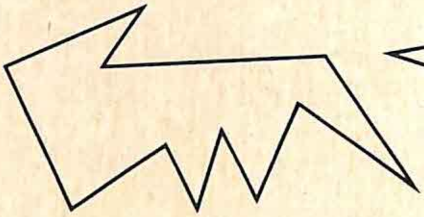
XV



XVI



XVII



XVIII

Exercise 8 Figures XIV to XVIII show various polygons. There is something special about these polygons as compared with those we have been dealing with. We want to find the sum of the interior angles of each.

a Look at XIV and XV. Can we find a point inside each so that the method of Exercise 6 will work?

b Can we do the same with XVI, XVII and XVIII?

c Can we use the method of Exercise 5 on these?

d Work out the sum of the interior angles for all five of these polygons.

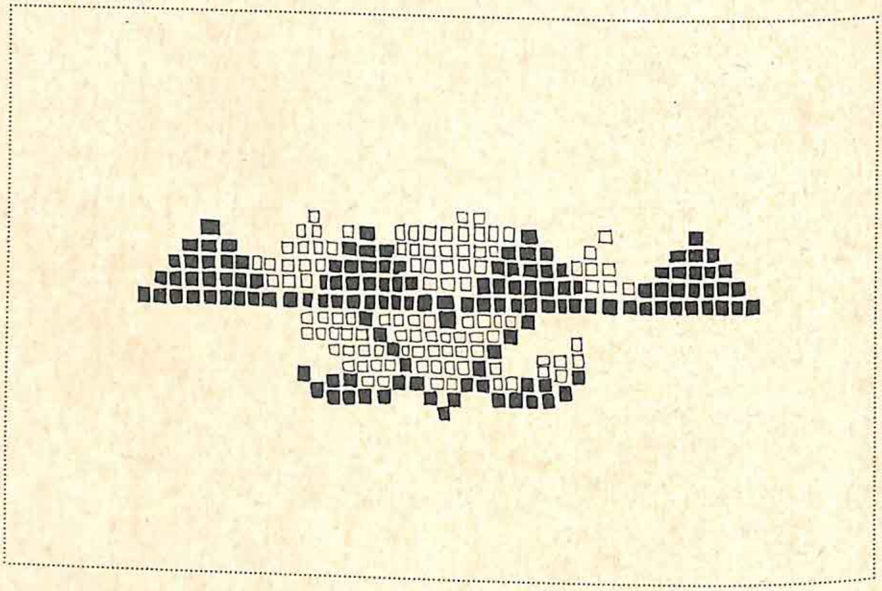
e If in figure XIV, we join the vertices A, B, C, D, E, F, G, H in order by straight segments, what sort of polygon is inside the figure? Use this method to work out the sum of the interior angles. Does it give the same results as the two previous methods?

Here is your list of polygons and their names.

number of sides	name
3	triangle, trigon
4	quadrilateral, tetragon
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	enneagon
10	decagon
11	hendecagon
12	dodecagon

Section 26 **Pavements and surfaces**

The illustration shows a floor of a Roman house. Have you ever seen one? There are many in this country, some in museums and others still where the Romans left them. How were they put together?



Exercise 1 **a** Have you ever seen a large floor covered with tiles, all of the same shape? Look around you for examples: the school hall, the pavement outside and so on. Make a list of as many as you can with sketches to show how the shapes are put together.

b Look for other examples of flat surfaces which are covered by repeating a shape many times. Again make a list with sketches. Look at the side of a house, at the pattern of the windows of a glass-sided office block for example.

c Can you think of other examples where a shape is repeated all over some surface?

The language spoken by the Romans was Latin. One of the Latin words for a tile was *tessera* from which we get our English word *tessellation*. When we cover a surface with repeated shapes or when we divide a surface up by repeated shapes, we say we are *tessellating* the surface.

Exercise 2 You have already met some of the shapes which we can use to tessellate a flat surface.

a Make a list of all the shapes you can find which will tessellate a flat surface. Partition your set of shapes into (i) regular and (ii) irregular shapes.

b Could you use the shapes of the following objects to cover a surface completely? (Beware: there may not be a simple *yes* or *no*!)

i pennies
ii threepenny pieces
iii nuts of the same size (the kind that fit bolts!)
iv copies of *Learning Mathematics I*
v end-flaps of 'Toblerone' packets
vi 8-sided postage stamps, e.g. the British Guiana 1d. magenta, 1856
vii 3-sided postage stamps, e.g. Cape of Good Hope green shilling, 1853

c Find out which, if any of the following regular shapes will tessellate a plane.

i octagon
ii pentagon
iii decagon
iv hexagon
v square
vi equilateral triangle
vii any other regular polygons you can find

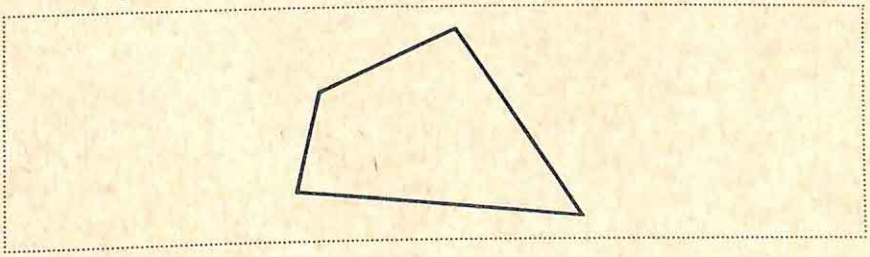
d Sometimes an interesting tessellation can be made with two simple polygons put together. What other regular polygon could be used with each of the following?

i regular octagon
ii regular pentagon
iii regular enneagon

e Some polygons which are not regular can tessellate a plane. Say which of the following will if used by themselves.

i isosceles triangles
ii scalene triangles
iii quadrilaterals

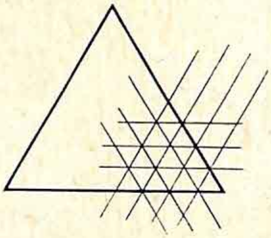
(For the last one you will probably want to cut out about a dozen identical quadrilaterals



and try tessellating with them.)

Exercise 3 **a** Draw the tessellation of a piece of plain paper by equilateral triangles. You will do this most easily in the following way, if all your lines are extended to the edge of the paper:

- i draw an equilateral triangle with a 4 in. base;
- ii mark $\frac{1}{2}$ in. units along the base;
- iii draw lines through these points parallel to the other sides of the triangle;
- iv through the intersections of the lines you have just put in, draw a set of lines parallel to the base of the original triangle.



b Examine your drawing carefully and then answer the following questions.

- i Can you find three families of parallel lines?
- ii Are all the triangles the same way up?
- iii How many triangles meet at any crossing point of three lines?
- iv Can you outline a regular hexagon?
- v Is there a tessellation by regular hexagons there too?
- vi Is there a tessellation by equilateral triangles other than the small ones?
- vii Is there a tessellation by hexagons bigger than the small ones you have already found?

c Still from your drawing, answer the following:

- i How many small triangles go to make up the smallest hexagon?
- ii How many small triangles go to make up the next sized hexagon?
- iii How many triangles in the next sized hexagon?
- iv Can you continue this sequence without looking at a drawing now?

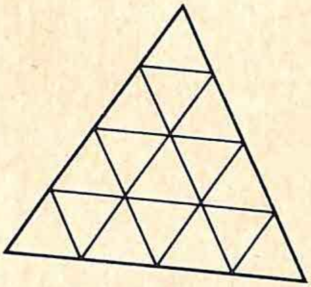
d

- i What is the angle of an equilateral triangle?
- ii What is the angle of a regular hexagon?

e

- i Can you draw a triangle on your tessellation 4 times as big as one of the small ones?
- ii Can you draw one 9 times as big?
- iii Can you draw one 16 times as big?
- iv What is this sequence of numbers and what is the next number?

f On your drawing draw a triangle with a base 4 times as big as the base of the small ones. It consists of 'layers':



Write down the numbers of small triangles in each layer in turn. It begins 1, 3, ... Do you recognize this sequence?

g i Can you draw a diamond equal to two triangles?

ii What can you say about the four sides of this diamond?

iii If a quadrilateral has four equal sides (not necessarily with the angles of your figures) it is called a *rhombus*. Can you find rhombuses on your diagram equal to two triangles of each size?

iv Can you draw trapeziums made of 3, 8 and 15 small triangles?

v Explain how this last sequence is made up and what is the next number.

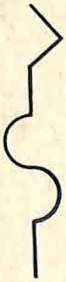
vi Can you draw a trapezium consisting of 12 small triangles?

vii Can you use this to start another sequence of numbers?

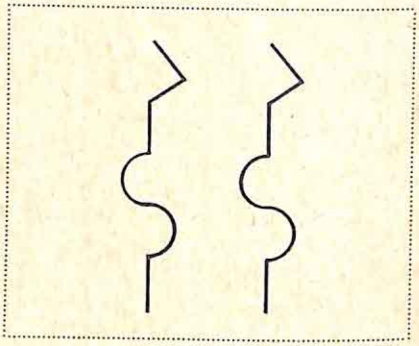
h We have only found a few of the interesting things about this tessellation. See how many more things you can discover for yourself.

We have been tessellating with regular and irregular polygons. We have seen that we cannot tessellate with circles, but it does not follow that some curved shapes will not work. This is one way we can make up shapes.

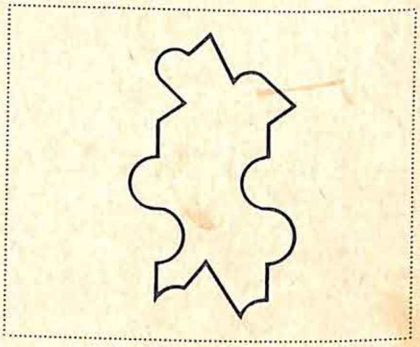
We begin with a line any shape:



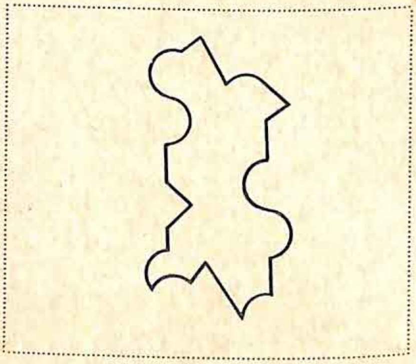
We repeat it:



We join the top and then repeat that shape on the bottom:



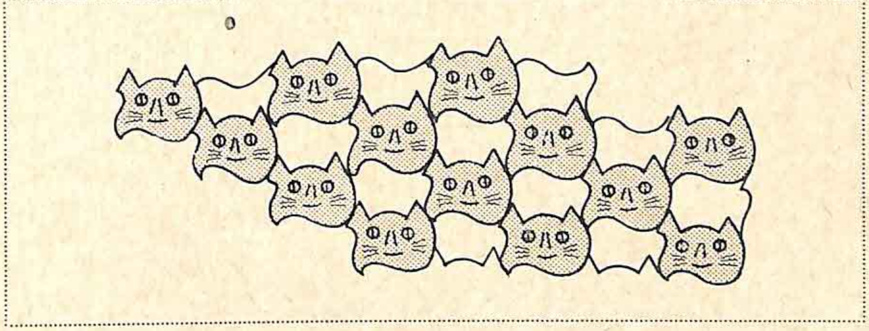
This shape will then tessellate a plane. We can vary the shape like this:

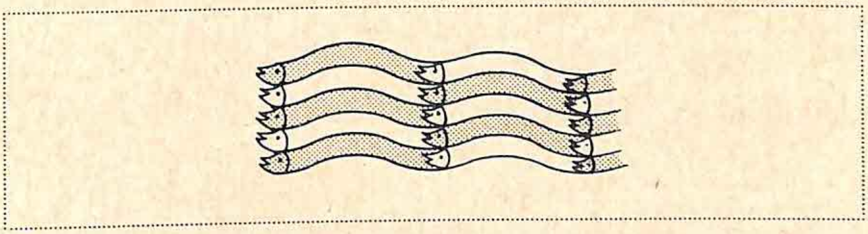


Make sure you can see what we have done and how we shall be able to tessellate with the shape.

Exercise 4 **a** Construct some shapes of your own which will completely cover a plane when they are repeated.

b If you are very good at this you may even be able to make your shapes look familiar:





Try doing something like this for yourself.

The amount of any surface covered by any simple closed curve (or polygon) is called the *area* of the figure. Each of the shapes you have been drawing has an area. If we wish, as we often do, to give the measure of this area, we need suitable units. A *unit* is merely some standard which we all agree to use, and it is usually chosen for the greatest possible convenience. When you were comparing the sizes of the various triangles you drew on your tessellated plane, you were really using area. You may have said 'This triangle is 16 times as big as one of the small ones'. Here your unit was *one small triangle*. We could all agree to use these triangles as measures of all our areas, or we could change them to triangles with a one inch base and use those. If we did this, we could then say that an area was 'so-many one-inch triangles in size', or, better, 'so-many triangular inches in area'.

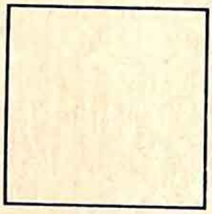
Exercise 5 **a** Say how big each of the following equilateral triangles is in 'triangular inches'.

i a 4 in. triangle
ii a 7 in. triangle
iii a 1 ft. triangle
iv a 1 in. triangle

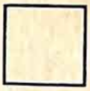
b i Would there be 12 triangular inches in 1 triangular foot?
ii How many triangular feet would there be in a triangular yard?

c If we used a metric system instead, we would use 'triangular centimetres'. How many triangular centimetres would there be in a triangular metre?

As you see from the last question, the triangular inch can be rather a difficult unit to use. What would be a more convenient unit? Clearly, some rectangular shape, in fact a square, will be the best. We will take as the unit for the measurement of area the 1 in. square, or in the metric system the 1 cm. square. Let us call the area of a 1 in. square *1 sq. in.* and that of a 1 cm. square, *1 sq. cm.*

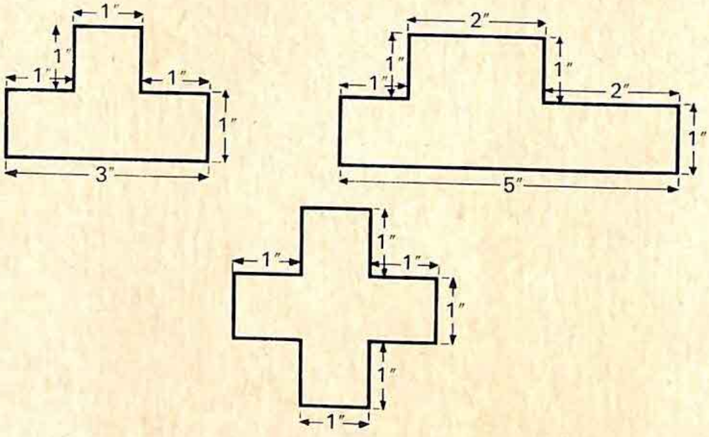


A 1 in. square
area: 1 sq. in.



A 1 cm. square
area: 1 sq. cm.

Exercise 6 **a** Sketch the following three figures full size on squared paper. Do they consist of a whole number of squares? What are their areas?



b Can you tessellate a plane with the following?

i rectangles

ii any of the figures in (a)

c i Draw sketches to show three different ways of tessellating with rectangles 9 in. by 4 in.

ii Draw sketches to show how any rectangle can be used in a herringbone pattern to tessellate a plane. (Look at a wood-block floor for a hint here.)

d Why is a brick wall put together the way it is and not in the simplest way? After all, it would be possible to build without using half bricks if the simplest sort of tessellation pattern were used.

e From a piece of squared paper, cut out a 6 in. square.

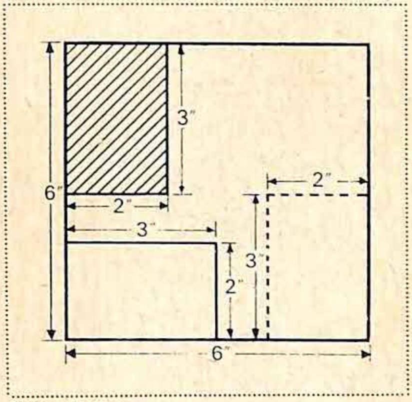
i What is its area?

ii Cut out a rectangle like the one shaded in the diagram. What is the area of this rectangle?

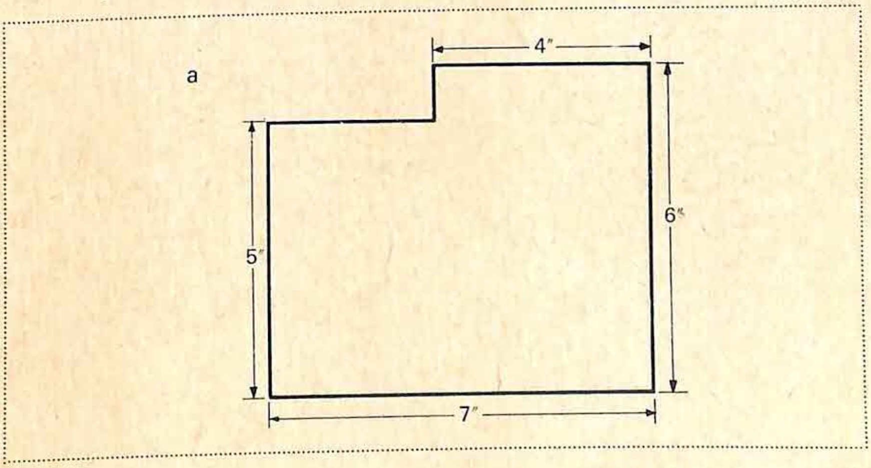
iii Cut out a second rectangle like the one not shaded in the diagram, and then a third like the dotted one. Now complete the diagram by cutting out a fourth rectangle of the same size, making the diagram symmetrical. What is the total area you have cut out?

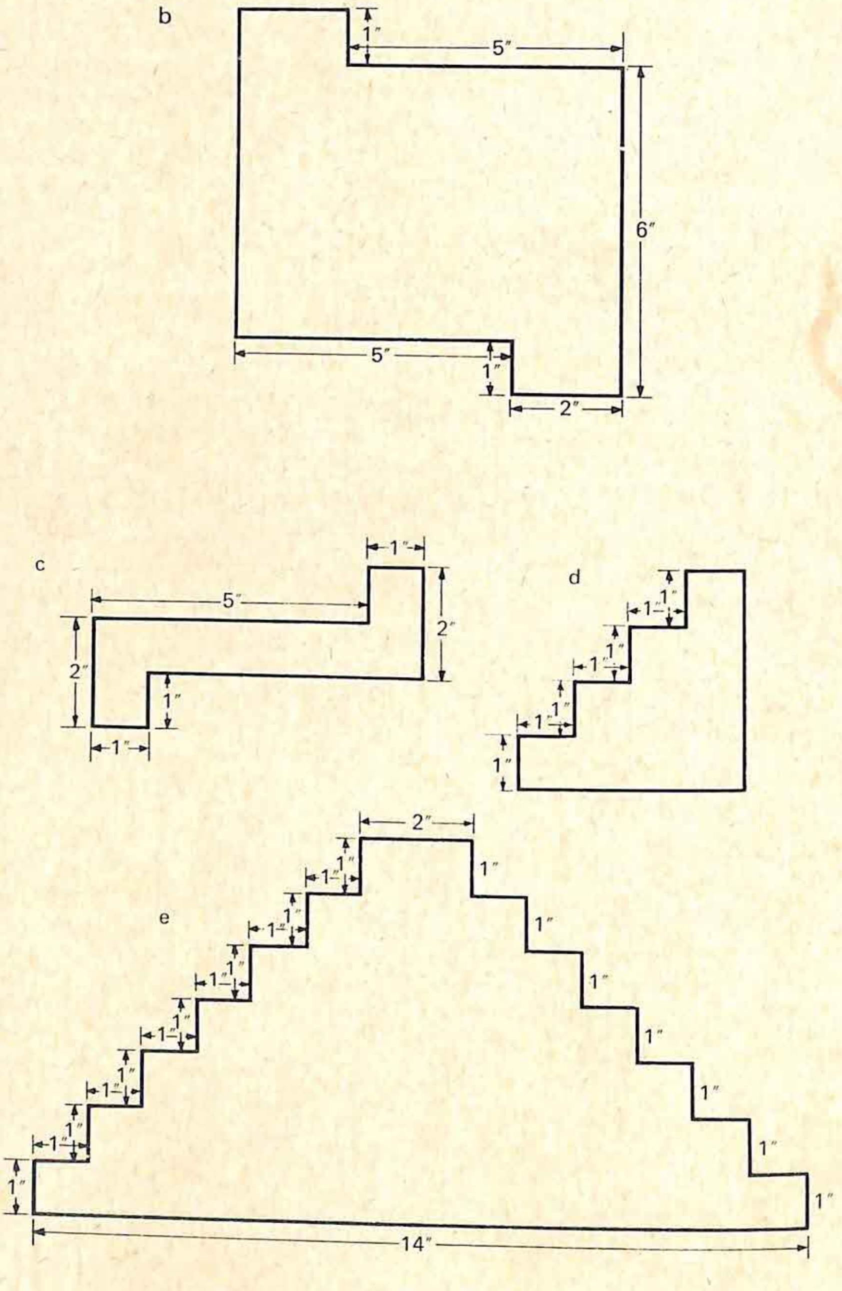
iv What is the area left?

v What is the sum of the answers to iii and iv?



Exercise 7 Find the areas of the following figures in square inches, using the simplest method you can see. All the angles are right-angles.




b
1″
5″
6″
5″
1″
2″
c
1″
5″
2″
2″
1″
1″
d
1″
1″
1″
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e
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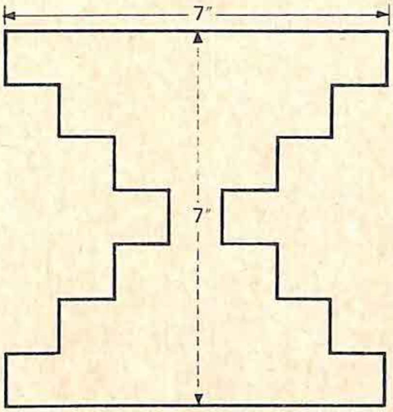
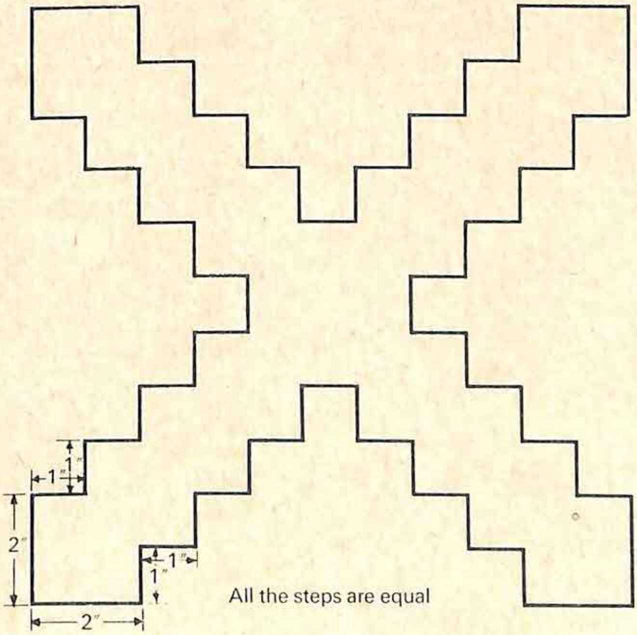
1″
1″
1″
14″

f



All the steps are equal

g



All the steps are equal

Exercise 8 **a** Would it be reasonable to give the area of an assembly hall floor in sq. in.? Explain.

b If you were going to buy a piece of carpet or linoleum from a roll, would you expect to order the quantity you wanted in either square inches or square feet? Explain.

c How many 1 in. squares will make a 1 ft. square?

d How many square inches are there in a square foot?

e How many 1 foot squares will be needed to tessellate a 1 yard square?

f How many sq. ft. are there in a sq. yd.?

Exercise 9 **a** If I tessellate a rectangle 6 in. long by 4 in. wide in sq. in., how many squares will there be?

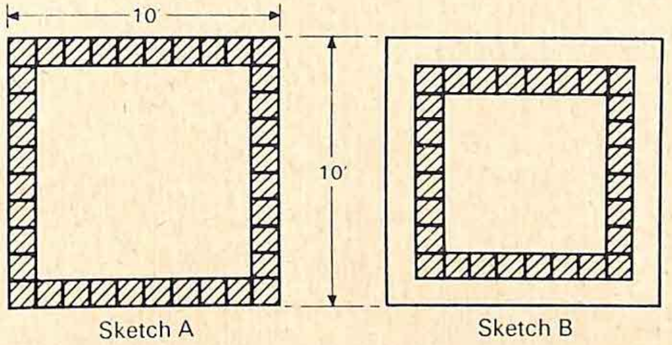
b Complete this table of areas of rectangles:

	length	breadth	area	(unit)
i	5 in.	7 in.		sq. ins.
ii	5 in.	6 ft.		sq. ft.
iii	2 ft.	x ft.		sq. ft.
iv	3 in.	y in.		sq. ins.
v	a ft.	2 ft.		sq. ft.
vi	a ft.	2 in.		sq. ft.
vii	a in.	6 in.		sq. ins.
viii	l units	b units		sq. units

c Can you think of other words which can be used for (i) length and (ii) breadth in certain circumstances? (Think of the ends of boxes, of walls, the edge of a book, of your ruler and so on.)

d I intend to make a chessboard from a piece of plywood and some 2 in. squares of black adhesive plastic. How big will the finished board be? What area of plastic shall I be using? If I can only buy the plastic in multiples of 1 in. lengths (the width being 18 ins.), how little can I buy? If the plastic costs 3s. 9d. per yard, how much is the piece going to cost me?

e Mr Smith is a handyman and he intends to lay the floors in his newly built house. He can get some plastic tiles 1 ft. square and he decides to floor the dining area with them. He would like a floor in two colours, black and red. One pattern he thinks of is illustrated in sketch A and another in sketch B.



The shaded area consists of a single row of black tiles in each. In sketch B, the black tiles are surrounded by a single line of red ones. If black tiles are a little dearer than red ones, which method will be cheaper? Find the following answering 'A' or 'B' or 'neither'

i a method which uses 40 black and 64 red tiles
ii a method which uses 36 black and 87 red tiles
iii a method which uses 36 black and 64 red tiles
iv a method which uses 56 red and 38 black tiles
v a method which uses 26 black and 30 red tiles
vi a method which uses 72 red and 28 black tiles
vii What are the lengths and breadths of the red rectangles in A & B?
viii What is the difference between the two red areas in B?

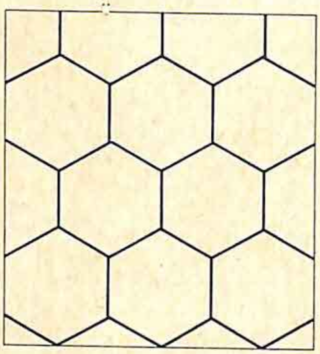
Exercise 10 a Each brick in a wall has one face showing. Find the approximate area of the face including a layer of cement on top and at one end.

b By counting bricks find the approximate area of the end of a house or of a brick wall, including the mortar.

Exercise 11 A path is made of hexagonal concrete blocks whose edges are 1 ft. long.

a By drawing to scale on graph paper, find the approximate area of the top face of one block.

b The illustration shows a path made of these blocks. It is 3 blocks wide and 4 blocks long. What is the approximate area of a path 4 blocks wide and 10 blocks long?



Exercise 12 Select the correct answers for the following:

a A plane can be tessellated with regular:

i hexagons ii pentagons iii octagons iv triangles

b A regular hexagon can be divided up into

i 8 ii 12 iii 6 iv 4 equilateral triangles.

c The angle *x* in the figure below is

i 45° ii 30° iii 60° iv 120°

d The angle *y* in the figure below is

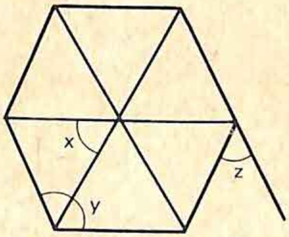
i 45° ii 30° iii 60° iv 120°

e The angle *z* in the figure below is

i 45° ii 30° iii 60° iv 120°

f Area is

- i length times breadth
- ii lb
- iii inches times inches
- iv a measure of a region of a surface
- v a number of square inches



A regular Hexagon

g There are in a square foot

- i 48 sq. in.
- ii 144 sq. in.
- iii 1 sq. in.
- iv 1728 sq. in.

h In a sq. yd. there are

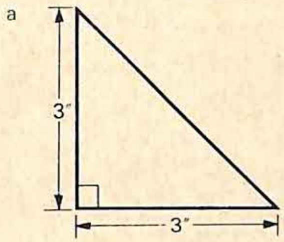
- i 9 sq. ft.
- ii 3 sq. ft.
- iii 36 sq. in.
- iv 1296 sq. in.

i In a sq. m. there are

i 10 sq. cm.
ii 100 sq. cm.
iii 1000 sq. cm.
iv 10000 sq. cm.

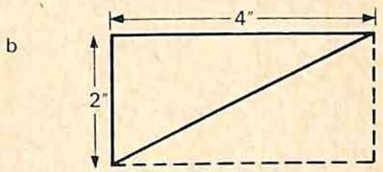
Exercise 13 Here are some more areas for you to work out. You will find answers to alternate questions printed at the end of this section.

a Draw this right-angled triangle accurately on squared paper.



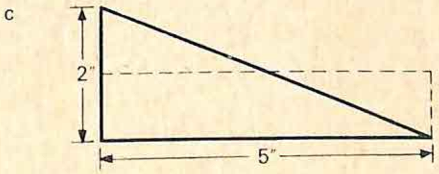
i What is its area?
ii How does your result compare with that of a 3 in. square.

b i Try this one by tessellation.

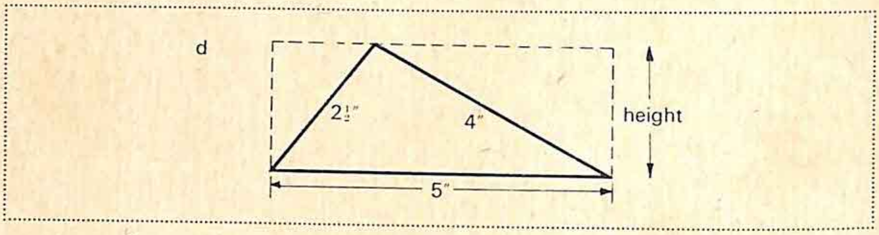


What is its approximate area?
ii The dotted line will help you to *calculate* the area exactly. Find it.

c Can you use the method suggested by the dotted lines to calculate the exact area of this triangle?



d Draw this triangle on $\frac{1}{10}$ in. graph paper



i Can you estimate the area?

ii Compare the estimation with the area of the surrounding rectangle.

e Draw another triangle with a 5 in. base and the same height as the previous one.

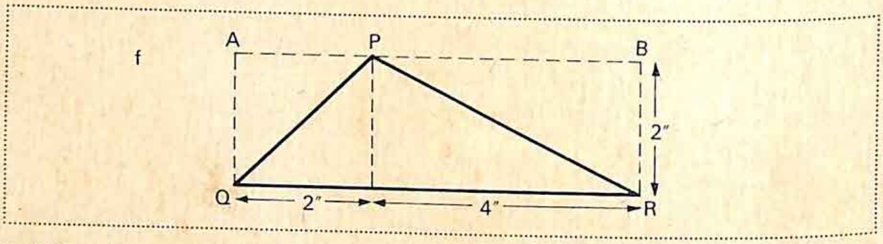
i What is its approximate area?

ii How does this compare with the area of the surrounding rectangle?

iii Do you think there might be a rule about this?

iv Try checking your rule with more drawings.

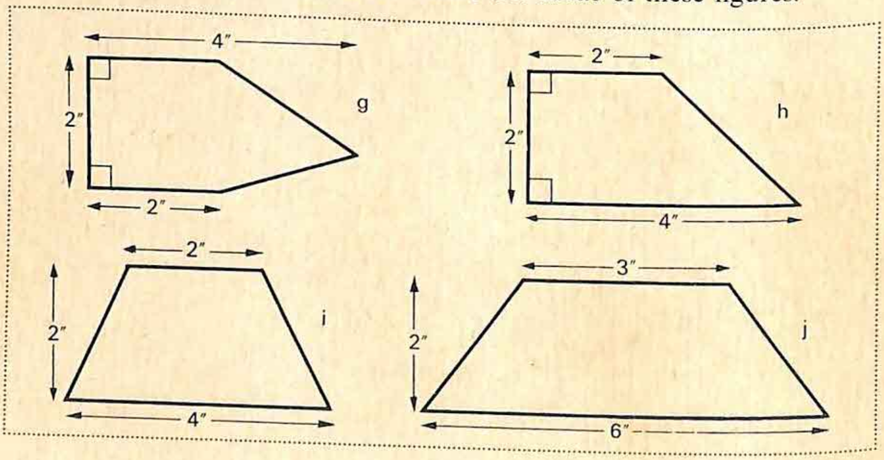
f

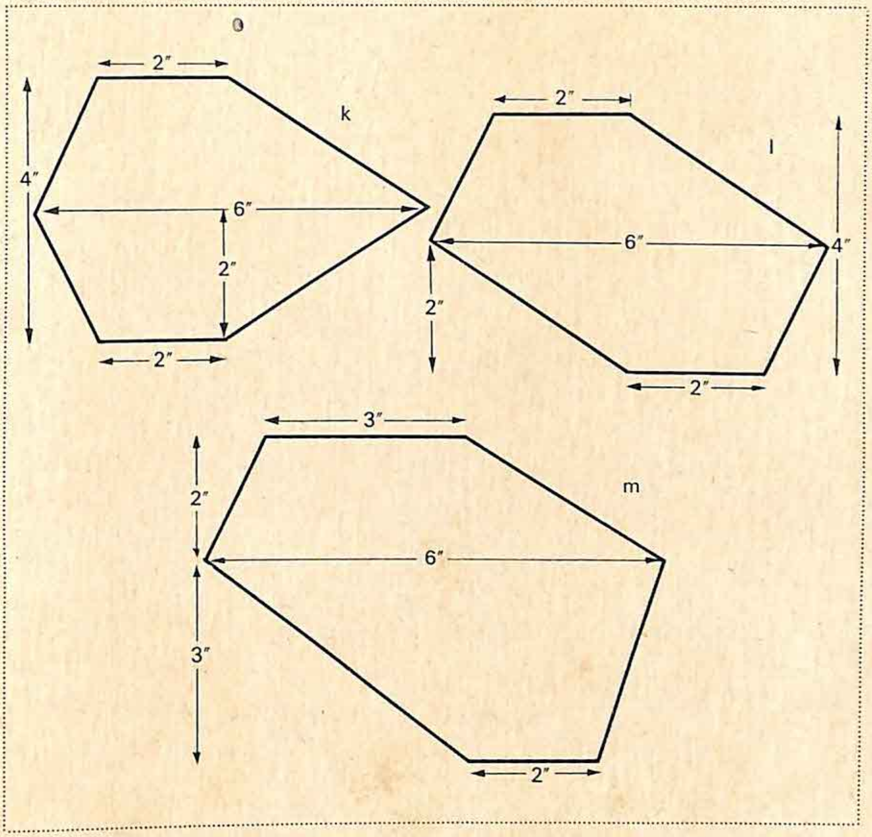


i What is the area of this triangle?

ii If QR remains the same length, what happens to the area of triangle PQR if we move P to other positions along AB?

g – m Use any methods you wish to *calculate* the areas of these figures.

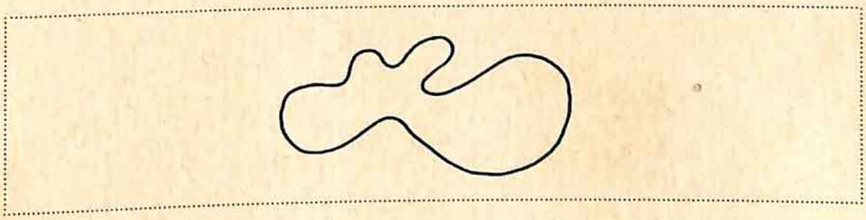




n Can you suggest a rule for the calculation of the area of a *trapezium*? (h, i and j are trapeziums.)

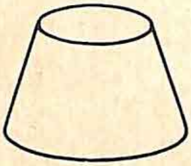
Exercise 14 You can now see how we can find the approximate area of irregular and curved shapes by drawing on graph paper.

a Find the approximate area of a shape such as this:

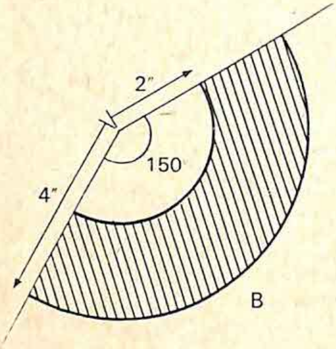


b Find the approximate area of the sole of your shoe.

c Find the approximate area of a circle of 2 in. radius.



A



B

d i Has this lamp shade A an area?
  ii Is it proper to speak of the area of a surface which is not plane?
  iii Draw the figure B on squared paper and cut out the shaded area. What is its approximate area?
  iv Can you make the shape of the lamp-shade A out of the paper cut-out?

Answers to Exercise 13

a $4\frac{1}{2}$ sq. in.
c 5 sq. in.
f 6 sq. in.
h 6 sq. in.
j 9 sq. in.
l 16 sq. in.

Section 27 **Digits in their places**

We talked about the place system of notation in Section 2. We saw that in the denary system, places to the left of the units were ten times bigger at each step. Suppose we now continue the places to the right. Now reading from the left the places stand for ... hundreds, tens, units and (if we continue dividing by 10) tenths, hundredths, thousandths, ... To mark the point where the new numbers are, we put a dot, ·. (On the Continent, they use a comma and in America a full stop.) We call the point a *decimal point* and the numbers which follow it a *decimal fraction.* Decimal fractions are another way of expressing rational numbers.

etc.	THOUSANDS	HUNDREDS	TENS	UNITS		TENTHS	HUNDREDTHS	THOUSANDTHS	etc.
	9	8	7	6	·	5	4	3	

Nine thousand, eight hundred and seventy-six point five, four, three.

Exercise 1 **a** 31·24 means $3 \times 10 + 1 + 2 \times \frac{1}{10} + 4 \times \frac{1}{100}$.
Rewrite these in full:

i 5·6
ii 14·8
iii 25·67
iv 67·89
v 143·3
vi 0·471

b 0·24 means $\frac{2}{10} + \frac{4}{100}$ which works out to $\frac{24}{100}$.
Work out the following, each as a single fraction.

i 0·43
ii 0·27
iii 0·341
iv 0·8742

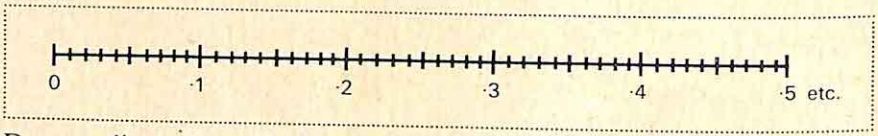
c Any common fraction which has an equivalent fraction with 10 in the denominator, can be expressed as a decimal fraction. $\frac{1}{2} = \frac{5}{10}$ so $\frac{1}{2} = 0{\cdot}5$.
Express each of the following as decimal fractions.

i $\frac{1}{5}$
ii $\frac{3}{5}$
iii $\frac{4}{5}$
iv $\frac{3}{2}$
v $\frac{7}{2}$
vi $\frac{22}{5}$

d Any common fraction which can be expressed as an equivalent fraction with 100, or 1000, or 10,000 etc. in the denominator can also be written as a decimal fraction. Convert each of these.

i $\frac{7}{20}$
ii $\frac{9}{50}$
iii $\frac{21}{25}$
iv $\frac{76}{125}$
v $\frac{47}{250}$
vi $\frac{431}{500}$

Exercise 2 Compare $\frac{2}{10}+\frac{3}{10}=\frac{5}{10}$ with $0{\cdot}2+0{\cdot}3=0{\cdot}5$; and compare $\frac{2}{10}+\frac{5}{100}+\frac{4}{10}+\frac{3}{100}=\frac{6}{10}+\frac{8}{100}$ with $0{\cdot}25+0{\cdot}43=0{\cdot}68$. This suggests that decimal fractions add up just like ordinary numbers. They subtract like ordinary numbers too.



a Draw a line segment 10 in. long and mark the end-points 0 and 1. Mark points on it to represent 0·1 and 0·2 etc.

Divide each of the segments between these numbers into ten parts. Now use your number line to plot the following points:

i 0·56
ii 0·43
iii 0·92
iv 0·05

b Now try addition on the line in the same way as you did with fractions.

i Step off 0·25 beyond 0·43. Does the result come to 0·68 as it did when we tried adding the other way?
ii Add 0·56 to 0·43
iii $0{\cdot}56+0{\cdot}25$
iv $0{\cdot}92-0{\cdot}43$
v $0{\cdot}56-0{\cdot}43$
vi $0{\cdot}43-0{\cdot}25$

c Check each of these results by arithmetic.

d Add up the following pairs of numbers. Alternate answers are at the end of the section.

i 4·3, 3·2
ii 1·7, 0·52
iii 6·4, 4·3
iv 5·6, 2·3
v 4·5, 3·6
vi 7·9, 5·6
vii 9·1, 0·9
viii 4·6, 2·4
ix 4·33, 2·17
x 5·26, 4·43
xi 5·62, 3·45
xii 6·45, 3·55
xiii 0·63, 0·084
xiv 6·3, 0·84
xv 0·063, 0·0084

e Subtract the second number from the first in each of the pairs in (d).

f i What is the result if we multiply $\frac{2}{10}$ by 10?
ii What do we get if we multiply 0·2 by 10?
iii What is $100\times\frac{27}{100}$?
iv What is $100\times0{\cdot}27$?
v What is $100\times0{\cdot}45$?
vi What is $1000\times0{\cdot}371$?
vii Find $100\times2{\cdot}93$
viii Work out $100\times6{\cdot}8$
ix Calculate $1000\times18{\cdot}56$

g i $27 \div 10$ iv $270 \div 1000$ vii $0{\cdot}01 \div 10$
ii $2{\cdot}7 \div 10$ v $64 \div 10$ viii $0{\cdot}056 \div 100$
iii $2{\cdot}7 \div 100$ vi $16{\cdot}4 \div 100$ ix $8{\cdot}042 \div 1000$

When we multiply 23 by 6 we actually say $6 \times (20+3)$ or, shorter, $6(20+3) = 6 \times 20 + 6 \times 3$. This rule is important because it allows us to do 'long multiplication'.

Exercise 3 **a** Explain another way of writing each of these, apart from working them out.

i $6 \times (4+7)$ iii $6 \times (2{\cdot}3)$ v $0{\cdot}6 \times 0{\cdot}63$
ii $7 \times (\frac{1}{7} + \frac{3}{14})$ iv $0{\cdot}7 \times 0{\cdot}54$

b Work out each of those expressions by the shortest method you can.

c Multiply each of these numbers first by 2 and then by 0·2

i 0·4 iii 0·5 v 0·9
ii 0·7 iv 0·1 vi 0·2

d Multiply the numbers first by 3 and then by 0·3

e Work out:
i $0{\cdot}2 \times 0{\cdot}02$ ii $0{\cdot}4 \times 0{\cdot}02$ iii $0{\cdot}6 \times 0{\cdot}06$

Now	$\frac{4}{10} \times 2 = \frac{8}{10}$	$\frac{4}{10} \times \frac{2}{10} = \frac{8}{100}$
so	$0{\cdot}4 \times 2 = 0{\cdot}8$	$0{\cdot}4 \times 0{\cdot}2 = 0{\cdot}08$
similarly	$\underbrace{0{\cdot}5 \times 3}_{\downarrow} = \underbrace{1{\cdot}5}_{\downarrow}$	$\underbrace{0{\cdot}5 \times 0{\cdot}3}_{\downarrow} = \underbrace{0{\cdot}15}_{\downarrow}$
number of digits following dec. pts.	1 1	2 2

We know that	$\frac{41}{100} \times 2 = \frac{82}{100}$	$\frac{41}{100} \times \frac{2}{10} = \frac{82}{1000}$	$\frac{41}{100} \times \frac{2}{100} = \frac{82}{10,000}$
so	$0{\cdot}41 \times 2 = 0{\cdot}82$	$0{\cdot}41 \times 0{\cdot}2 = 0{\cdot}082$	$0{\cdot}41 \times 0{\cdot}02 = 0{\cdot}0082$
and	$\underbrace{0{\cdot}52 \times 3}_{\downarrow} = \underbrace{1{\cdot}56}_{\downarrow}$	$\underbrace{0{\cdot}52 \times 0{\cdot}3}_{\downarrow} = \underbrace{0{\cdot}156}_{\downarrow}$	$\underbrace{0{\cdot}52 \times 0{\cdot}03}_{\downarrow} = \underbrace{0{\cdot}0156}_{\downarrow}$
number of digits following dec. pts.	2 2	3 3	4 4

Consider

$$\begin{aligned} \underbrace{0{\cdot}21\times 0{\cdot}34}_{4} &= 0{\cdot}21(0{\cdot}3+0{\cdot}04)\\ &= 0{\cdot}21\times 0{\cdot}3+0{\cdot}21\times 0{\cdot}04\\ &= 0{\cdot}063 \quad +0{\cdot}0084\\ &= \underbrace{0{\cdot}0714}_{4} \end{aligned}$$

number of digits following dec. pts.

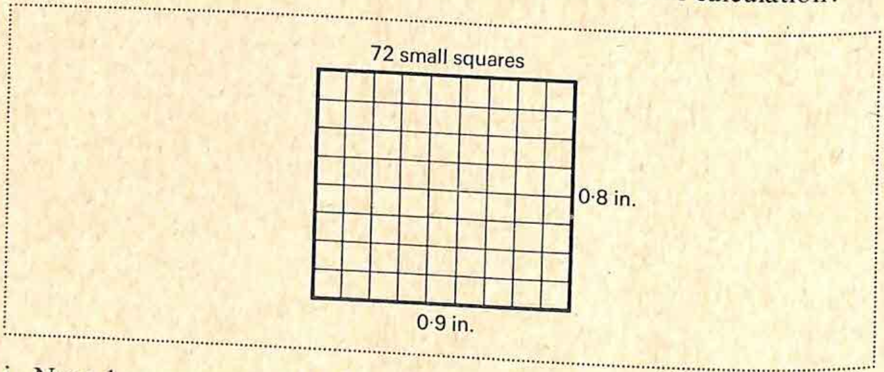
Can you see a rule for multiplication now? Can you see how to work out, say, $0{\cdot}14\times 0{\cdot}29$? If you cannot, then try 21×34 and compare this answer with the answer to $0{\cdot}21\times 0{\cdot}34$ which is done for you above. The only real problem is, as you see, where to put the decimal point.

Exercise 4

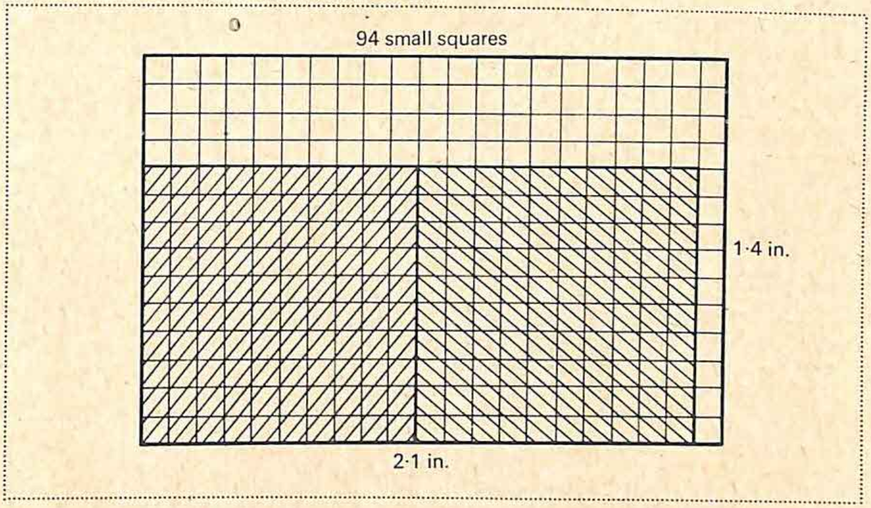
a $0{\cdot}21\times 0{\cdot}24$
b $0{\cdot}44\times 0{\cdot}21$
c $0{\cdot}27\times 0{\cdot}64$
d $2{\cdot}1\times 1{\cdot}4$
e $0{\cdot}21\times 0{\cdot}14$
f $3{\cdot}7\times 1{\cdot}8$

Exercise 5 For this exercise you will need $\frac{1}{10}$ in. graph paper.

a i Draw a rectangle on the paper 0·5 in. long and 0·3 in. wide.
ii How many small squares go to make up this rectangle?
iii How many small squares go to make up one square inch?
iv What is the area of one small square?
v What is the area of your rectangle?
vi Is this the area which you would have expected from calculation?

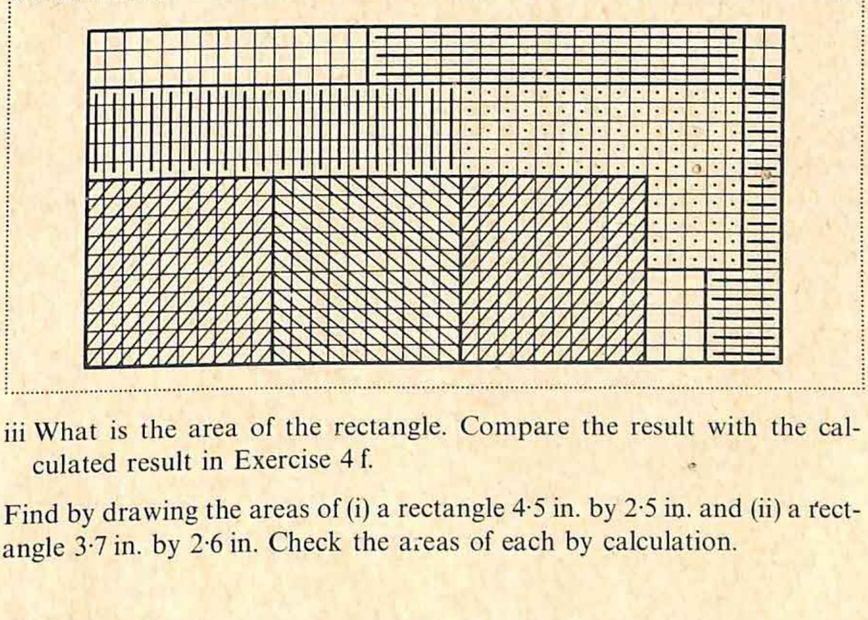


b i Now draw a rectangle 2·1 in. by 1·4 in.
ii Shade in the two whole square inches and count the small squares. What is the area of the rectangle?
iii Compare the area with the calculation in Exercise 4 d.



c i Draw a rectangle 2·3 in. by 1·2 in.
ii Find its area and check by calculation.

d Draw a rectangle 3·7 in. by 1·8 in.
ii Shade in the complete 1 in. squares. Now continue to shade the small squares in sets of 100. How many squares are left unshaded?



iii What is the area of the rectangle. Compare the result with the calculated result in Exercise 4 f.

e Find by drawing the areas of (i) a rectangle 4·5 in. by 2·5 in. and (ii) a rectangle 3·7 in. by 2·6 in. Check the areas of each by calculation.

Exercise 6 Multiply the pairs of numbers in Exercise 2 d. You will find some of the answers at the end of the section.

Look at this sequence of calculations to see where they are leading.

A $1{\cdot}2-0{\cdot}3-0{\cdot}3-0{\cdot}3-0{\cdot}3=0$ so $1{\cdot}2\div 0{\cdot}3=4$
$12\div 3=$ $\qquad$ $1{\cdot}2\div 0{\cdot}3=$
$120\div 3=$ $\qquad$ $0{\cdot}12\div 0{\cdot}03=$

B $2{\cdot}4-0{\cdot}8-0{\cdot}8-0{\cdot}8=0$ so $2{\cdot}4\div 0{\cdot}8=3$
$24\div 8=$ $\qquad$ $2{\cdot}4\div 0{\cdot}8=$
$240\div 80=$ $\qquad$ $0{\cdot}24\div 0{\cdot}08=$

Exercise 7 **a** i What happens to the answer when we multiply both the number and its divisor by 10 or 100 or 1000, etc.?

ii What happens if we divide both the number and its divisor by 10 or 100 or 1000, etc.?

b Complete the following statements:

i $1{\cdot}0-0{\cdot}2-\qquad=0$ and $1{\cdot}0\div 0{\cdot}2=$
ii $3{\cdot}6-0{\cdot}6-\qquad=0$ and $3{\cdot}6\div 0{\cdot}6=$
iii $0{\cdot}1-0{\cdot}01-\qquad=0$ and $0{\cdot}1\div 0{\cdot}01=$
iv $2{\cdot}1-0{\cdot}07-\qquad=0$ and $2{\cdot}1\div 0{\cdot}07=$
v $10\div 2=$
vi $0{\cdot}01\div 0{\cdot}002=$
vii $100\div 20=$
viii $0{\cdot}36\div 0{\cdot}06=$
ix $0{\cdot}036\div\qquad=6$
x $3600\div\qquad=6$
xi $2{\cdot}1\div\qquad=30$
xii $2{\cdot}1\div\qquad=300$
xiii $2{\cdot}1\div\qquad=3$
xiv $0{\cdot}0021\div\qquad=30$
xv $0{\cdot}021\div\qquad=300$
xvi $0{\cdot}21\div\qquad=3$

Now look at the following sequences of ideas. (Read down the columns.)

A $12\div 3=4$
$120\div 3=40$
$1200\div 3=400$

B $12\div 3=4$
$12\div 0{\cdot}3=40$
$12\div 0{\cdot}03=400$

C $1{\cdot}2\div 0{\cdot}3=40$
$12\div 0{\cdot}3=$
$120\div 0{\cdot}3=$
$1200\div 0{\cdot}3=$

D $1{\cdot}2\div 0{\cdot}3=40$
$1{\cdot}2\div 0{\cdot}03=$
$1{\cdot}2\div 0{\cdot}003=$
$1{\cdot}2\div 0{\cdot}0003=$

Exercise 8 **a** Complete these statements:

i $12000\div 0{\cdot}3=$
ii $3{\cdot}5\div 0{\cdot}7=$
iii $35\div 0{\cdot}7=$
iv $350\div 0{\cdot}7=$
v $4{\cdot}9\div 0{\cdot}7=$
vi $490\div 0{\cdot}7=$
vii $49\div 0{\cdot}7=$
viii $2{\cdot}7\div 0{\cdot}9=$
ix $2700\div 0{\cdot}9=$
x $2{\cdot}7\div 0{\cdot}09=$
xi $2{\cdot}7\div 0{\cdot}009=$
xii $4{\cdot}9\div 0{\cdot}07=$
xiii $4{\cdot}9\div 0{\cdot}0007=$
xiv $49\div 0{\cdot}07=$

b When we talked about the equivalence of fractions, we made statements like this one: $\frac{1}{2} = \frac{10}{20} = \frac{100}{200} =$

Complete the following statements.

i $\frac{1}{4} = \frac{}{40} = \frac{}{400} = \frac{}{4000}$

ii $\frac{1}{3} = \frac{10}{} = \frac{100}{} = \frac{1000}{}$

iii $\frac{3}{8} = \frac{30}{} = \frac{}{100}$

iv $ = \frac{20}{100} = \frac{}{1000}$

c Can you say that 0·1 ÷ 0·2 = 1 ÷ 2 and that 1·2 ÷ 0·3 = 12 ÷ 3 and that 1·2 ÷ 0·03 = 120 ÷ 3? If so, why?

This gives you a method of dividing decimals. If a question is not easy to do, you change it to something else which is easier. The important thing is to make sure that the answers to the two questions are the same. It will always be enough to make the divisor into a whole number. For instance 5·6 ÷ 0·08 is best done as 560 ÷ 8 and 5·6432 ÷ 0·08 is best changed to 564·32 ÷ 8.

Exercise 9 Alter the following questions so that division would be easier.

a Divide 3·6 by 0·6

b Divide 4·64 by 0·02

c 73·2 ÷ 0·003

d 494·9 ÷ 0·7

e 0·04 divided into 64·16

f 0·0002 divided into 0·004

Exercise 10 a Work out the divisions of Exercise 9. The answers may be found at the end of this section.

b Work out the following:

i 0·414 ÷ 0·03
ii 2·37 ÷ 0·3
iii 2·42 ÷ 0·11
iv 0·341 ÷ 0·011
v 23·075 ÷ 0·005
vi 8·181 ÷ 0·006
vii 56·1 ÷ 0·03
viii 2·97 ÷ 0·0011
ix 1·188 ÷ 0·022
x 270·549 ÷ 0·00009

Answers to Exercise 2

d i 7·5 v 8·1 ix 6·50 xiii 0·714
iii 10·7 vii 10·0 xi 9·07

e i 1·1 v 0·9 ix 2·16 xiii 0·546
iii 2·1 vii 8·2 xi 2·17

Answers to Exercise 6

i 13·76
iii 27·52
v 16·20
vii 8·19
ix 9·3961
xi 19·3890

Answers to Exercise 10 a

i 6
ii 232
iii 24,400
iv 707
v 1604
vi 20

Section 28 **Filling space**

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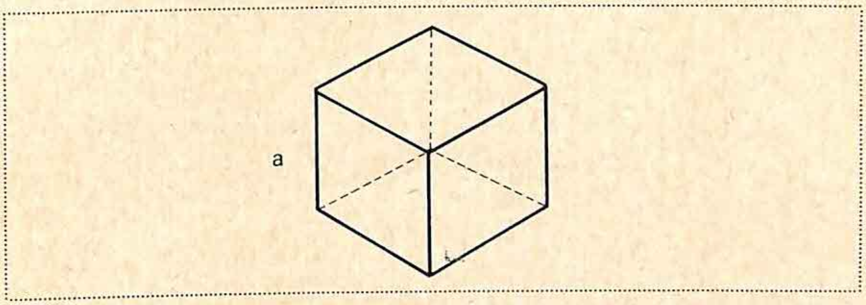
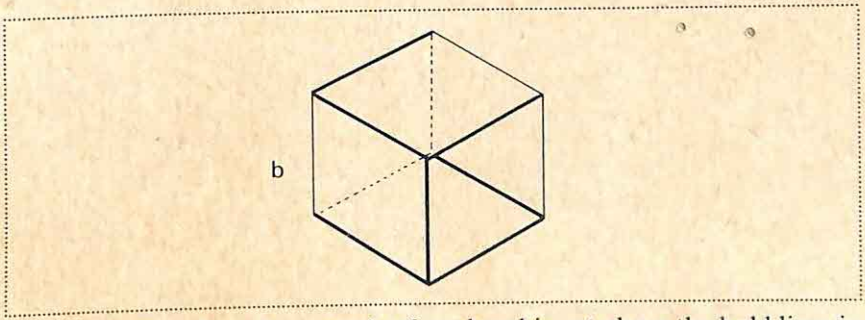
Exercise 1 The sketch shows a familiar *solid*, that is a three-dimensional shape which occupies space. It is called a *cube*: all its angles are right angles and all its edges are equal.



a Look at the drawing, think about the shape and give the answers to the following questions.

i How many *faces* has the cube?
ii How many *vertices* has the cube?
iii How many *edges* has the cube?
iv Let F stand for the number of faces, V for the number of vertices and E for the number of edges. Work out F + V − E.

b



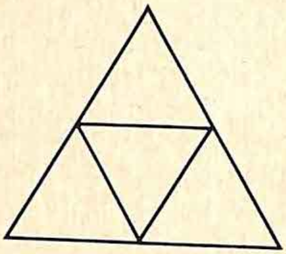
i Imagine that the cube is made of card and is cut along the bold lines in this sketch. Draw the shape it opens out to.
ii Can you find another shape which will fold up to give a cube?
iii These shapes you have drawn are called *nets* for the cube. Both nets could be used to make a card cube. Cut one of them out in drawing paper and fold it into the shape of a cube. Use tape to join the edges together.

iv If you had to do this again, could you put tabs onto some of the edges to make sticking possible without tape? How many tabs would you need and where on the net would you put them?

c If you had many cubes all of the same size, could you fill space with them and leave no gaps between them?

As you will realise the word *solid* is used in mathematics in a different way from the ordinary use of the word. In chemistry, a solid is something which is neither a liquid nor a gas; in everyday life a solid thing is firm and unyielding; it means something quite different in connexion with gold watches! In mathematics the word is used for three-dimensional figures which occupy space: a line the shape of a bed-spring is not a solid in this way, nor is a surface the shape of a saucer. A solid too has at least one surface. You will find the answers to Exercise 1 at the end of the section. Check them before you go on.

Exercise 2 **a** Here is a net. Copy it onto card or stiff paper, put on the tabs you need, cut it out and fold it into a solid.



All edges are 2″

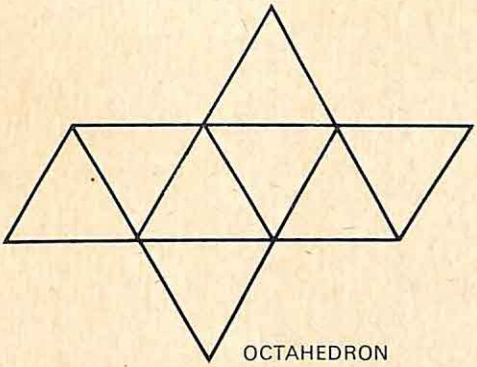
b i Work out F + V − E for this solid.

ii The word *hedron* refers to the faces. Why do you think this solid is called a *tetrahedron*?

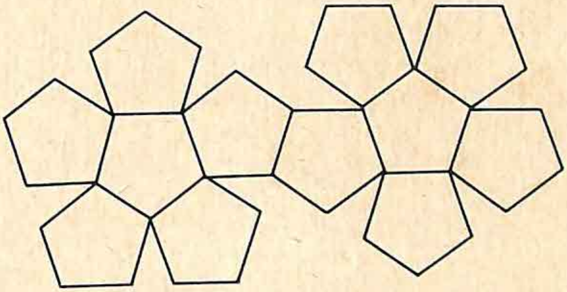
iii It is in fact a *regular tetrahedron*. Why is it regular?

Exercise 3 There are other solids called pentahedron, hexahedron, heptahedron, octahedron, enneahedron, decahedron, hendecahedron, dodecahedron, polyhedron. What do you think these words mean? There is one other name you may meet – it is the icosahedron, the twenty-faced solid.

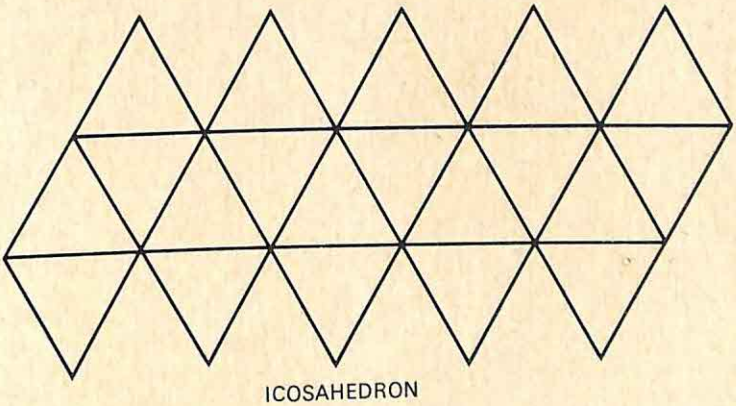
Exercise 4 Here are four other nets for you to use for making solids.



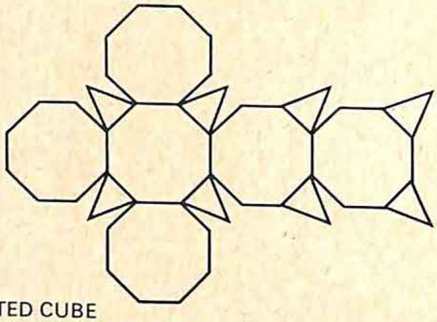
OCTAHEDRON



DODECHAHEDRON
All the faces are regular pentagons the angles of which are 108°



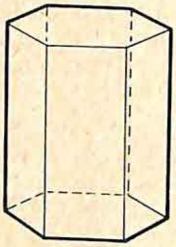
ICOSAHEDRON



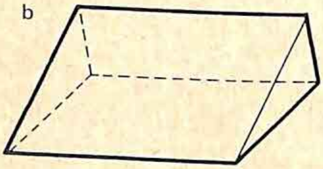
TRUNCATED CUBE
Regular octagons and equilateral triangles

Exercise 5 Could you fill space completely with the solids illustrated or named here?

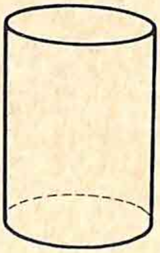
a



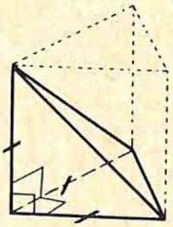
b



c



d

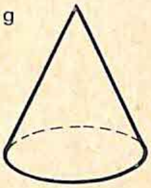


d (in this we have dotted in a second solid to help you)

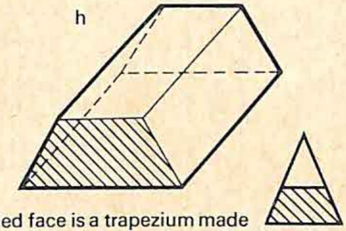
e Regular tetrahedrons

f Regular octahedrons

g



h



The shaded face is a trapezium made
by cutting off the top of an isosceles triangle

Exercise 6 **a** Bricks are usually said to be 9 in. long, 4 in. wide and 3 in. deep. Sketch your answer to each of the following, putting in the dimensions.

i Can you combine 4 bricks to form a block 36 in. long?

ii Can you rearrange the four bricks to make a block 16 in. long?

iii Can you use the four bricks to make a block 12 in. long?

iv Could you arrange 6 bricks to make a block 27 in. long and 8 in. wide?

v Arrange 10 bricks in a block 45 in. long and 6 in. wide.

vi Make a block 36 in. long and 8 in. wide with 16 bricks.

vii Make a block 12 in. long and 8 in. wide with 32 bricks.

viii Make a block 6 ft. long and 1 ft. deep with 64 bricks.

b i How many bricks would I need to build a wall 9 ft. long, 12 in. deep and 8 in. wide?

ii How many bricks would be needed for a wall 36 ft. long, 18 in. deep and 8 in. wide?

c i Measure some actual bricks to discover whether the measurements we have been using are strictly correct.

ii Do you think that with real bricks you could stack them in the exact way we have been describing? Explain.

iii If you built the walls referred to in (b), would you find that the number of bricks you have worked out would be correct? Explain.

iv Would the numbers you have given be maximum or minimum numbers?

v What is the maximum number of bricks there could be in a pile 90 ft. long, 80 ft. wide and 5 ft. high?

If we want to measure the amount of space inside a box, we work out how many blocks of a known size would fit inside it. This number gives us the *capacity* of the box. If we want to measure the size of a solid, we compare its size with a pile of standard blocks. The number of blocks gives us the *volume* of the solid. It is important that we all use the same standard block. It is best too that this be a cube. We may use the cubic centimetre, the cubic metre, the cubic inch, the cubic foot or the cubic yard. For very large volumes we might need the cubic kilometre or the cubic mile.

Exercise 7 Let us think about the 1 cm. cube.

a
- i How many cubic centimetres are there in a 1 cm. cube?
- ii How many cubic centimetres (c.c.) are there in a block 2 cm. long, 3 cm. wide and 1 cm. deep?
- iii What will be the volume of a single layer of these cubes 12 cm. long and 6 cm. wide?
- iv What would be the volume of a pile of these cubes 12 cm. long, 6 cm. wide and 5 cm. high?
- v What is the volume of a block 25 cm. long, 4 cm. wide and 2 cm. high?
- vi What is the volume of a block 34 cm. by 15 cm. by 3 cm.?
- vii Find the volume of a block 45 cm. by 17 cm. by 13 cm.
- viii Find the volume of a rectangular block 127 cm. long by 56 cm. wide by 29 cm. high.

b
- i What is the capacity of a box 14 cm. by 12 cm. by 15 cm. inside?
- ii What number of cubic centimetre blocks would go into a box 56 cm. by 89 cm. by 23 cm.?
- iii A box exactly holds 240 c.c. blocks. It is 10 cm. long and 8 cm. wide. How deep is it?
- iv A box will hold exactly 450 cubic blocks in 6 layers, 5 blocks wide. How many blocks long is it?
- v Does the size of the blocks in iv matter?
- vi 1200 cubic blocks are arranged in a pile which is 25 blocks long. Find all the possible combinations of height and width which can be obtained.

c How many 1 c.c. blocks form
- i a single layer 6 cm. long and 7 cm. wide?
- ii a double layer 6 cm. long and 7 cm. wide?
- iii three layers 6 cm. long and 7 cm. wide?
- iv n layers 6 cm. long and 7 cm. wide?
- v n layers p cm. long and 7 cm. wide?
- vi n layers p cm. long and q cm. wide?
- vii a block l cm. long, b cm. wide and h cm. high?
- viii If you have put in multiplication signs in vi, vii, viii above then rewrite them without these signs.

Exercise 8 Work out the following volumes and capacities. All the boxes and blocks referred to are rectangular in shape. You will find the answers to alternate questions at the end of this section.

a Volume of a block 7 in. long, 4 in. wide, 3 in. high.

b Volume of a block 9 ft. by 6 ft. by 7 ft.

c Volume of a block 8 yds. by $4\frac{1}{2}$ yds. by 3 yds.

d Capacity of a box 7 in. by $6\frac{1}{2}$ in. by 4 in.

e Capacity of a box $9\frac{1}{2}$ ft. by 2 ft. by 5 ft.

f Volume of a block $10\frac{1}{2}$ yds. by 4 yds. by 7 yds.

g Capacity of a box $5\frac{1}{2}$ in. by $5\frac{1}{2}$ in. by 4 in.

h Volume of a block $6\frac{1}{2}$ ft. by $6\frac{1}{2}$ ft. by 6 ft.

i Volume of a block $3\frac{1}{2}$ yds. by $3\frac{1}{2}$ yds. by $3\frac{1}{2}$ yds.

j Capacity of a box $4\frac{1}{2}$ in. by $4\frac{1}{2}$ in. by $4\frac{1}{2}$ in.

k Capacity of a box 2·5 cm. by 4 cm. by 3 cm.

l Volume of a block 5 cm. by 4·5 cm. by 6 cm.

m Volume of a block 12·2 cm. by 4 cm. by 5 cm.

n Capacity of a box 8·4 cm. by 3 cm. by 10 cm.

o Volume of a block 2·6 cm. by 3·5 cm. by 4 m.

p Volume of a block 2·8 cm. by 4·2 cm. by 5 m.

Exercise 9 **a** Suppose I build a 1 ft. cube using 1 in. blocks

i How many cubes will there be in each layer?
ii How many layers will there be?
iii How many cubes will be used altogether?
iv How many cubic inches are there in a cubic foot?

b How many cubic feet are there in a cubic yard?

c How many cubic centimetres are there in a cubic metre?

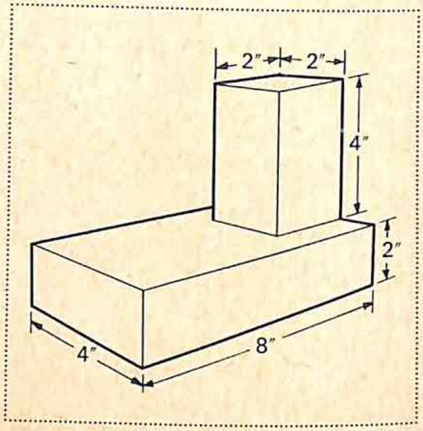
Exercise 10 **a** A child's wooden building brick is 2 in. each way.

i How many cubic inches of wood are there in each brick?
ii How many cubic inches of wood are there in a pile of bricks 12 in. high?

b The child built the 'church' shown here.

i How many bricks did the child use?

ii How many cubic inches of wood are there in the model?



c The *Choc Box* sweet shop in Polchester sells its own home-made chocolates and fudge. The fudge is cut into 1 in. cubes which are then packed into boxes a little more than 3 in. by 4 in. by 6 in. These boxes are sold for 12s. 6d. each. The boxes cost 6d. each. How much does each piece of fudge cost?

d A half pound packet of butter I have just bought measures $2\frac{1}{4}$ in. by $1\frac{1}{4}$ in. by 5 in.

i How many cubic inches of butter is this to the nearest whole number?

ii The packet was made up from a keg which contained almost exactly 784 cubic inches of butter. Was the keg

A 7 lb. B 14 lb. C 21 lb. D 28 lb. E 35 lb.?

Exercise 11 **a** I put 100 one-inch cubes into a box which is 10 in. long and 5 in. wide. How deep is the layer of cubes?

b If a box 4 cm. wide and 3 cm. deep contains 120 one-centimetre cubes when it is full, how long is it?

c 250 one-centimetre cubes make a layer 2 cm. deep over the bottom of a box 25 cm. long. How wide is the box?

d A 15-in long box contains 360 one-inch cubes in a 2-in. layer all over the bottom. How wide is the box?

e A rectangular pile is made up of boxes all of which are 1-ft. cubes. There are 210 boxes and the pile is 6 ft. wide. It is more than a foot long and high. How wide and high is the pile?

f I have two boxes. One is $1\frac{1}{2}$ ft. by 6 in. by $1\frac{1}{2}$ in. while the other has a capacity of 825 c.c. The length and breadth of the second box are just smaller than those of the first and are a whole number of centimetres.

The second box will not go inside the first. What are the dimensions of the smaller box?

g Make up four questions like (a) to (d) above for your neighbour to work out. You will find that you need the factors of some fairly large numbers to do it.

h Here is another puzzle for you. I have a box whose capacity is 385 cm. and whose sides are whole numbers of centimetres. No dimension is less than 5 cm. What are the dimensions of the box?

i Find all the possible sets of dimensions of a box of capacity 60 cu in. if all the sides are a whole number of inches.

j Do the same for a box whose capacity is 270 c.c. if all the sides are a whole number of centimetres.

k Do the same again for a box of capacity 240 c.c. if all the sides are a whole number of centimetres.

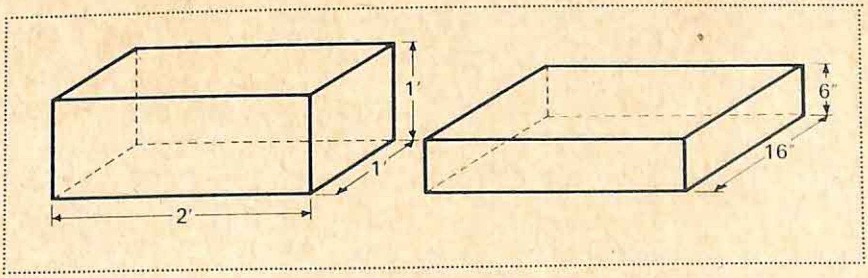
Exercise 12 **a** Take the contents of a full box of sand and tip it into a sandbag. The shape of the bag is different from that of the box. Is the volume of the sand the same?

b i If you had tipped the sand into a cylindrical oil drum, would the volume of the sand have changed?

ii How could you find the capacity of an oil drum practically?

c Do irregular shaped objects have volume and irregular shaped cans capacity?

d I have two fish tanks of different designs. I have only a broken tape measure with me for measuring, and it will only read up to 2 ft. 8 in.



I have measured the shorter tank and put its dimensions on to the drawing. I can measure the second tank in two directions but the tape measure is not long enough to measure its length. I do remember, however, that the first tank will empty into the second and fill it exactly. How long is the second tank?

e If it rains heavily one day and deposits $\frac{1}{4}$ in. rain all over a field which is 50 yds. by 20 yds., how many cubic yards of water have fallen?

f A gallon is the volume of 10 lb. of pure water at 62°F. This works out to be 277·42 cu. in. A tank is 10 in. long and 5 in. in each of the other directions.

i About how many gallons will it hold?
ii About how much will the contents weigh if it is full of water?

g About how many gallons will a cubic tank 1 ft. long hold?

h About how much will the water weigh in the fish tanks of question (d) above?

Exercise 13 **a** *Washo* is a new soap powder. It is packed in boxes 3 in. by 8 in. by 9 in. Unfortunately the boxes have to be kept the right way up. They are delivered in large crates 27 in. high. The base of the crates is 2 ft. 8 in. by 2 ft. 9 in. How many packets are there in each crate?

b A special large economy size of *Washo* is on sale. It contains 300 cu. in. of powder and costs one and a half times as much. Would you buy it in preference to the smaller size?

c In Queen Victoria's day, silver coins were made entirely of silver. Which would you rather have had (i) a tea chest full of half-crowns or (ii) the same chest full of sixpences?

d How would you set about finding the amount of sand (excluding the air) in a bowlful of sand?

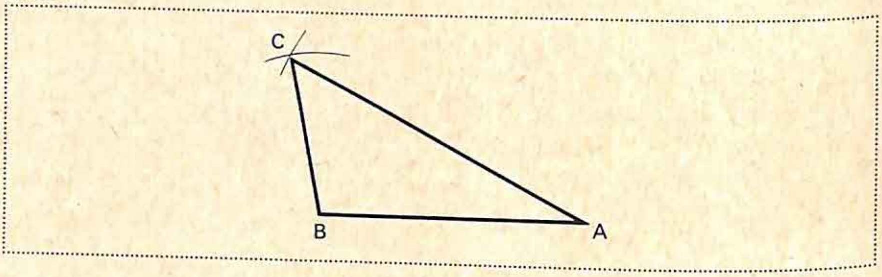
Answers to Exercise 8

a	84 cu. in.	**e**	95 cu. ft.	**i**	$42\frac{7}{8}$ cu. yds.	**m**	244 c.c.
c	108 cu. yds.	**g**	121 cu. in.	**k**	30 c.c.	**o**	36·4 cu. m.

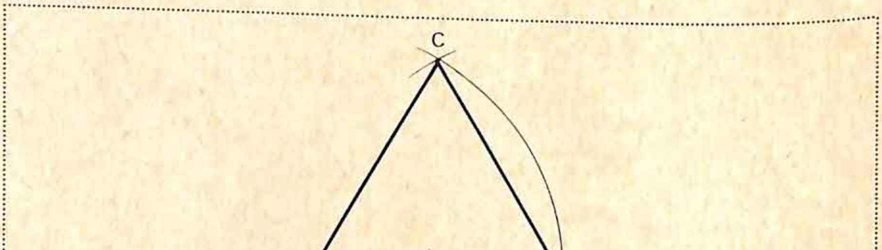
Section 29 **Drawing regular polygons**

In these exercises you will be constructing polygons with a straight-edge and a pair of compasses only. The measurements you will be given are only to help you to construct figures which are big enough to be fairly accurate and small enough to go onto your page.

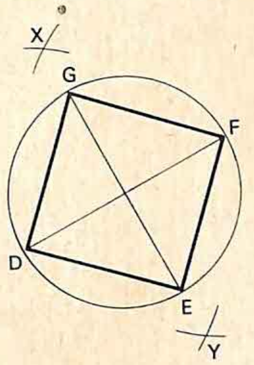
Exercise 1 The triangle. If you are given the three sides of a triangle you can construct it. Construct a triangle ABC in which AB is 7 cm. long, AC is 9 cm. and BC is 5 cm.



Exercise 2 The regular triangle (equilateral triangle). All the sides of this triangle will be 10 cm. Mark a point A near the left-hand edge of your page. With centre A and radius 10 cm. draw a large arc. With a centre at any point B on the arc, and with the same radius draw a small arc to intersect the large arc. Call this point C. △ABC is an equilateral triangle. Check that it is.

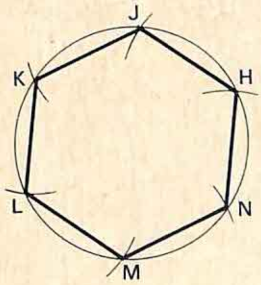


Exercise 3 The regular quadrilateral. (The square.) The diagonal of this square will measure 8 cm. Draw a circle of radius 4 cm. and draw in any diameter. Call the diameter DF. With centre D and any radius greater than the

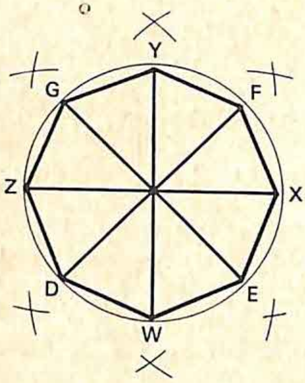


radius of the circle, draw an arc on either side of the diameter. Now keep the same radius and repeat this using F as the centre. Make the arcs intersect at points X and Y. Join XY. This line should pass through the centre of the circle. Now complete the square DEFG.

Exercise 4 The regular hexagon. This hexagon will have a longest diagonal of 10 cm. Draw a circle of radius 5 cm. From any point H on the circumference use your compasses with the same radius to step off the points J, K, L, M and N. You should now be able to step off one more radius which will bring you back *exactly* to H. Draw in the hexagon HJKLMN.



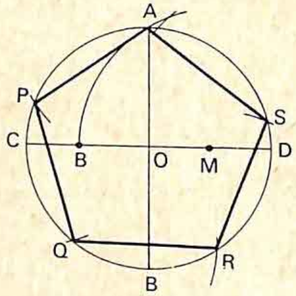
Exercise 5 The regular octagon. This octagon will have a greatest diagonal of 12 cm. Draw a circle of radius 6 cm. Now, as before construct the square DEFG inside it, but do not join up the vertices of the square. Now from each of the points D, E, F and G, with a radius about the same as that of the circle (not exactly though), draw two arcs. Make them intersect in pairs. Join opposite intersections and complete the octagon DWEXFYGZ.



Exercise 6

The regular pentagon. This one will be easier on 1 inch squared paper. If you use graph paper, ignore the fine lines.

Draw a circle radius 2 in. with its centre at the point where two lines intersect. Draw two diameters at right angles. Call the centre of the circle O and the two diameters AB and CD. Mark M, the mid-point of OD.

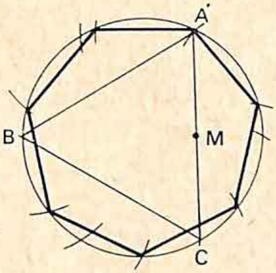


With centre M and radius equal to MA, draw an arc from A to cut CO in B. Take a radius AB on your compasses and starting at A, step off this radius round the circle until you come back to A. Call these points P, Q, R and S. Complete the pentagon APQRS.

Exercise 7

The regular heptagon. This construction is not accurate really. The error in the length of the sides of the figure will be rather less than one-and-a-half thousandths of an inch!

Draw a circle radius about 2 in. on plain paper. Step off the radius round the circle as with the hexagon. Ignore alternate arcs and draw the equilateral triangle ABC. Now find the mid point M of the side AC. Take a radius equal to AM on your compasses and step it off round the circle. It will give the vertices of a nearly regular heptagon. Complete the heptagon.



The construction of regular polygons you have been doing (except for the last) can be made as accurate as you please. The errors are yours and your instruments'. It is a strange fact that only some of the regular polygons can be drawn in this way with only a pair of compasses and a straight edge, without measuring any sides or angles.

Here is a table which shows which can be done accurately with up to 100 sides.

					51				
	12		32						
3									
4		24	34			64			
5	15							85	
6	16								96
	17								
8				48		68			
10	20	30	40		60		80		

Section 30 **Illustrating statistics**

If you have never been to Stoke-on-Trent, does it tell you much about the size to say that the population is 271,800 people? You really need to compare the population with some other towns to understand. Here is a table:

Stoke-on-Trent	271,800	Kingston upon Hull	300,800
Leicester	281,200	Newcastle upon Tyne	275,100
Nottingham	312,600	Bradford	287,000

Exercise 1

a Do you think the population figures are exact? If not, what has been done to them?

b What is the difference in population size between the largest and the smallest of these six towns?

c Would you be right if you said they were all about the same size? How do you know?

d Does it help comparison if you give the figures to the nearest 10,000?

e Does it help still more if you give them to the nearest 100,000? Work them out to see.

f If you give the population of a town as 300,000 to the nearest 100,000, what is the smallest the population could have been?

g What is the largest population which would be given as 300,000 to the nearest 100,000?

h In what way would it be true to say these towns are all about the same size?

From this exercise we have learned two things: one is that to understand figures we often have to compare them with other figures; the other is that 'rounding off' can help.

Man	Height
Jack Black	5 ft. 8 in.
Bill Brown	5 ft. $10\frac{1}{2}$ in.
Wilf Gray	5 ft. 11 in.
John Smith	5 ft. $7\frac{1}{2}$ in.
Fred White	5 ft. 9 in.

Exercise 2 **a** What happens if you give all of these figures to the nearest foot?

b Are all the men about the same size?

c If so, would this be useful information for a tailor?

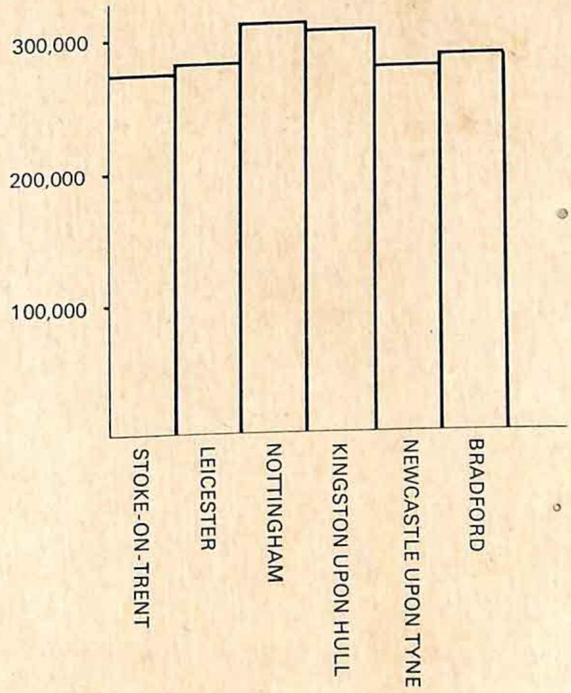
d What happens to the figures if you give the height to the nearest six inches?

e Does this tell you any more than before?

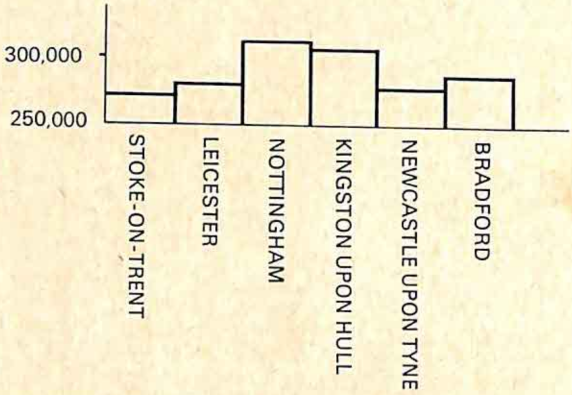
As we can see 'rounding-off' does not help in this case. It is better to leave the heights as they are if we are comparing the heights of these men. If on the other hand we compared European men with pygmy men, then it would be much more useful to say that the Europeans were about 6 ft. tall while the pygmies were about 3 ft. tall. Thus we could get a useful comparison. Whether we round off or not depends upon the kind of comparison we are making.

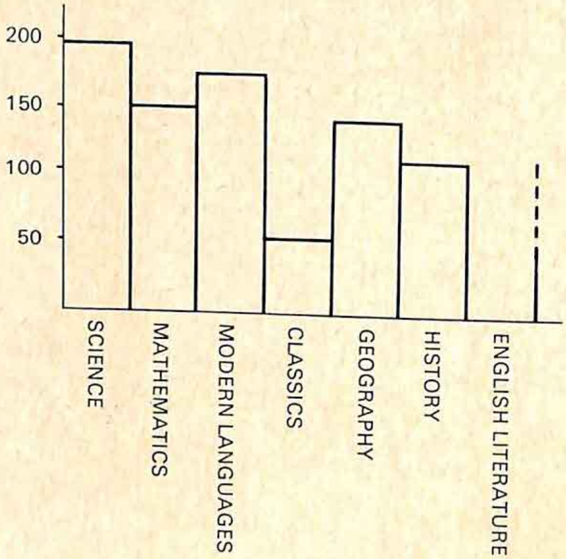
It sometimes helps us in making a comparison if we can make a picture of it. There are several kinds of picture we can make, and the rest of this section deals with some of them.

Here is a column graph of the populations in the first paragraph.



As you see the feet of the columns are all at the zero of the scale. The only part where there is a variation is at the top. We could show only the top part to show differences. But is it misleading? Does it not appear that Kingston upon Hull is about twice as big as Newcastle upon Tyne?





The diagram is a graph showing the numbers of books placed in a school library during 1965.

Exercise 3 Look at the drawing carefully and answer the following questions:

a In which subject were most books added?

b In which subject were fewest added?

c How many times more mathematics books than classics books were added?

d How many geography books were added?

e How can you find out how many books on modern languages were added?

f What was the total number of books added?

g Do you think your answer to (f) was exact?

h Can you say whether more money was spent on science books or history books?

i Are the 'bars' in the diagram actually numbers? Explain.

j What do you think we mean by 'scale' in this connexion?

k Is it likely that the numbers represented are exactly correct? What has been done with the correct numbers?

l If we doubled the height which represents 200 books, could we represent the numbers more accurately? Why?

m If 156 English literature books were also added, how long would the bar have to be on the diagram above to represent this?

Exercise 4 **a** Draw a graph to show the heights of a number of people in your school. You could select the people in several different ways. Here are some suggestions:

- i A group of your friends.
- ii One member, selected from the register, of each class in the school. (You could take, say, the fourth one on the register of each class.)
- iii The members of two teams (say hockey and basketball) who play for the school.
- iv Choose a boy and a girl of exactly the same age, and do this, say, five times over.

b Draw a graph to show the numbers of birds which land on the school playground or field during morning break on five successive days. It would be specially interesting to do this twice, with a three- or six-month interval between the two.

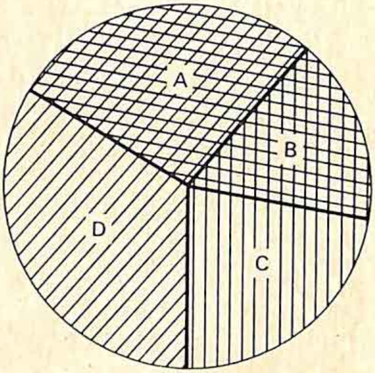
c i Choose five books from the school library and, from the records find out how many times each has been borrowed during a year.

ii Use the same five books and this time discover how long each was borrowed for and work out how long each was out of the library during the year.

Draw graphs to show these facts.

d Take a *small* dictionary and count the number of words beginning with each letter (or select a group of letters, say, A, D, E, M, P, Q, X, Z.) Draw a graph to show these comparisons. If your class has more than twenty-six members, then one person could count each of the letters and you could get all the facts.

e Open this book at the section about measuring (Section 19) and count the number of times I have used the words *centimetre*, *inch*, *foot*, *time*, and *length*. Draw a graph.



A sixth form contains 90 boys. Forty of these are in the Upper Sixth. Twenty of the Lower Sixth boys take Arts subjects. This 'pie' diagram illustrates this information and a good deal more as well. The circle is cut up to represent the only divisions of the Sixth Form into Lower and Upper, Arts and Science. In each form there are more boys taking science than arts subjects. Find out what each sector represents.

Exercise 5 **a** What does the whole circle represent?

b How many degrees is the angle representing the Upper Sixth?

c How big should be the angle used to illustrate the Lower Sixth?

d Identify the parts of the circle representing the Upper and Lower Sixth.

e Which sector represents the Lower Sixth Arts?

f Which sector represents the Upper Sixth Science?

g How many boys are there in each of the divisions of the Sixth?

h Would you notice a difference of one boy in the numbers represented in, say, the Upper Sixth Science group if you did not measure the angles?

Exercise 6 A boy receives £3 for his birthday. He plans to use it in the following way:

i One year's subscription to a magazine which costs 1s. 3d. per month;
ii 2s. 6d. on chocolate;
iii 12s. 6d. on a swimming bath season ticket;
iv the rest in the bank.

Draw a pie diagram to illustrate the uses to which he plans to put his money. Shade each sector differently and give a key to your shading.

Exercise 7 **a** What do you think might be the value of diagrams such as those in Exercises 3, 4, 5 and 6?

b Are impressions gained from the diagrams entirely trustworthy?

c In Exercise 3 the diagram does not show all the books added. Would a pie diagram be suitable for showing these same facts?

Exercise 8

Town	Population	Town	Population
Codsall	6,749	Walsall	117,836
Dudley	61,748	Willenhall	32,317
Tipton	38,091	Wolverhampton	150,385

The table shows a few towns in Staffordshire and their population sizes. Suppose you want to compare these facts by means of a bar diagram;

a Which of the following scales could be used?

i 1,000 of population represented by 1 in. of bar.
ii 10,000 of population represented by 1 in. of bar.
iii 100,000 of population represented by 1 in. of bar.
iv 100,000 of population represented by 1 cm. of bar.
v 10,000 of population represented by 1 cm. of bar.

Give reasons for your choice.

b Would you represent these numbers exactly? Could you, even if you used the largest of the suggested scales?

c What alteration would you want to make to the figures?

d Copy out the table making the alterations you suggest.

e Draw a bar graph using a scale you chose in (a).

f If you were 'reading' your diagram, what parts of the original information would you *not* find?

Exercise 9 **a** This table gives the distances of a number of towns from London. The distances are measured along the shortest road routes. Arrange them in order of size starting with the shortest and draw a suitable diagram to compare the distances. Draw also a map to show the towns with straight lines radiating from London to all of them.

	Miles		Miles
Hereford	133	Bristol	116
Dover	72	Shrewsbury	153
Norwich	111	Birmingham	110
Newcastle upon Tyne	274	Stoke-on-Trent	147
Gloucester	105	Inverness	531

b These are the distances of some Scottish towns from Edinburgh. Do precisely the same thing with these, including the drawing of a map.

	Miles		Miles
Aberdeen	117	Inverness	158
Fort William	133	Perth	44
Glasgow	44	Stranraer	125

c From each of your maps obtain the straight-line distances of these sets of towns from London and Edinburgh respectively. Arrange the towns in the same order as before and draw new graphs. Compare them to see if there is any great difference in the relative distances.

Exercise 10 Here is a table of sunrise and sunset for a year. The figures change somewhat from year to year, so do not trust this table too far! (The figures in any case apply to London.)

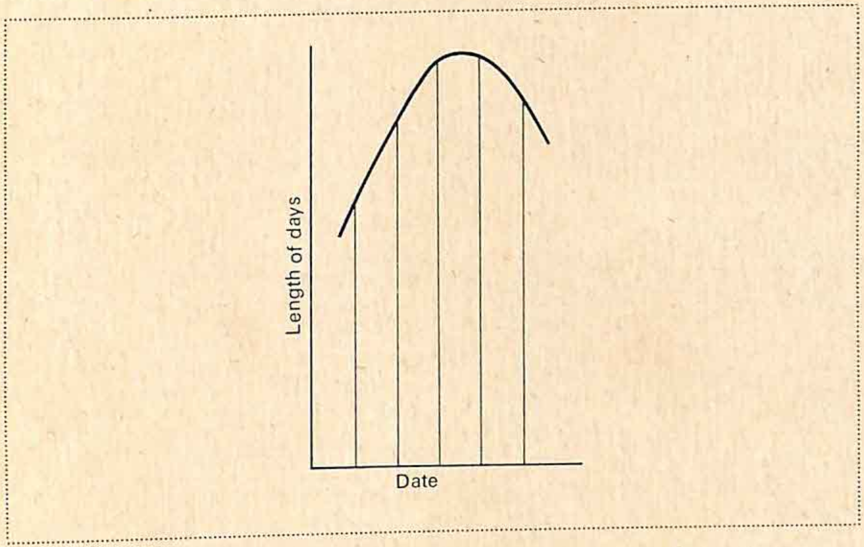
1st day of	Sunrise a.m. GMT.	Sunset p.m. GMT.
April	5.33	6.35
May	4.33	7.22
June	3.46	8.11
July	3.48	8.19
August	4.21	7.50
September	5.17	6.41
October	6.2	5.37
November	6.49	4.37
December	7.47	3.53
January	8.6	4.0
February	7.42	4.45
March	6.50	5.36

a Why do you think British Summer Time is introduced?

b Draw a graph to show the way the length of the day changes from month to month.

c i Do you think that your graph is the best kind? Can it tell you about the length of the day half-way through a month? Does that last phrase have any meaning?

ii Suppose you simply put in vertical lines instead of columns, and then joined the tops of these lines by a *smooth curve* (see the sketch) would this be better?



iii If you think it would be better, can you explain why? Why could we not do this with the other graphs we have drawn?

Exercise 11 These two tables show typical days in Fred's life. Draw suitable graphs to show how his day is used.

a In the week and

b on Saturdays.

c Can you find a way of using a column graph to show a good comparison between weekdays and Saturdays?

In all three graphs use the following categories:

i play
ii work
iii sleep
iv eating, washing, dressing, etc.

Weekday

7.30 a.m.	get up
8.30	leave for school
9.00	
	school
12.30	
	lunch
1.00 p.m.	
	play games
2.00	
	school
4.00	
	clubs
4.30	
5.00	arrive home, start tea
5.30	finish tea, start homework
7.30	finish homework
	free time
10.00	go to bed

Saturday

8.00 a.m.	get up
8.45	
	go out with friends
12.15	
	lunch with family
1.00 p.m.	
	go out again
5.00	
	tea
5.30	
	homework
7.00	
	amuse before bed
10.00	

Reinforcement sets

Set R1
Patterns in numbers

For this exercise use squared paper at least 20 squares wide and 15 squares deep. Copy the table begun below so that the number '10' is in the top right hand corner of your page. Complete the table up to 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	etc.						

Now answer the questions below:

a Reading across the rows we 'count in ones'. What are you counting in down the columns? Put a green ring round the numbers of the sequence beginning 10, 20, etc.

b Put a red ring round the numbers beginning 9, 18, 27 etc., that is counting in nines. What do you notice about the pattern they make?

c If you draw a line through the sequence 10, 20, 30 etc. and another through 9, 18, 27 etc. What do you notice? We will call the lines 'rays'.

d All the lines parallel to your tens ray count in tens. Is there a similar rule about nines?

e Can you draw a similar ray for counting in eights, and another for sevens? Can you carry on? Does the parallels rule hold?

f Where on the diagram would you put the number 0?

g Produce your rays to the left, off the present diagram. Should there by any numbers along these rays? Put some of them in until you have enough to see how to carry on and complete this part of the table.

h Can you now put in your rays for two and three?

i What will happen if you produce your rays again off the new diagram?

j Can you find rays for counting in elevens, twelves, etc.?

k Can you find another position for the number 0?

l Can you find rays for counting in 31's and 32's?

Set R2
Games with powers

a Make a table of the squares and cubes of the numbers 1 to 10; it is begun for you below.

A Number	1	2	3	4	5	6	7	8	9	10
B Square	1	4	9	16	25	.	.	.	.	.
C Cube	1	8	27	64	.	.	.	.	.	.

Now rewrite this table leaving in only the 'units' digit of each number in the square and cube lines.

A	1	2	3	4	5	6	7	8	9	10
B	1	4	9	6	5	.	.	.	.	.
C	1	8	7	4	.	.	.	.	.	.

Do you notice a pattern? Extend your tables up to 20 to see if the pattern continues. Also write the lines for fourth, fifth, etc. powers, using only the last digit each time. (Do you need to work out the whole of each number?) Carry on up to the ninth power. How many patterns can you find in the numbers?

b Write down a sequence of odd numbers beginning with 1. Add up i the first four terms, ii the first five terms iii the first six terms. What do you notice? Extend your additions both ways, to two and three terms and on beyond six terms. How are the results related to 1, 2, 3, 4, 5 etc.?

c Work out the squares of the odd numbers. Can you continue the pattern begun below?

$1^2 = 1$
$3^2 = 2+3+4$
$5^2 = 3+4+5+6+7$
$7^2 = 4+5+\ldots$ etc.
$9^2 = \ldots$ etc.
etc.

Set R3
Rounding off

An acre is a measure of area: an area of 4840 sq. yds. is called an acre.

a A plot of land is 50 yds. wide and just under 100 yds. long. About what area is it?

b A field is roughly square and its area is about 2 acres. About how long is it?

c Mr. Smith's car is a Mini-Minor and it uses 4 gallons of petrol to go 177 miles. The petrol he uses costs 4s. $10\frac{1}{2}$d. per gallon.

i About how many miles per gallon does his car do?

ii If he wants to go on a journey of 300 miles from a garage and has only £1 notes in his wallet, what is the smallest amount of money he can manage with?

iii If he uses a petrol pump where he can buy exactly £1's worth of petrol and his tank is empty, about how far can he travel on £1?

d An exercise is going to need manilla card but we want to use the card with as little waste as possible. Say whether the following can be cut from a six-inch square card; if so, how? (Sketches will help you).

i Cut out right angled triangles, one with a base 5 in. and height 4 in., the other with base 6 in. and height 6 in. Could you also cut out a 6 in. length of card 1 in. wide from the piece? If not, could you cut a 5 in. piece $\frac{1}{2}$ in. wide instead?

ii Two rectangles 2 in. by 4 in., a square 4 in. by 4 in. and a square 2 in. by 2 in.

iii One rectangle 2 in. by 5 in., one 1 in. by 2 in., one 1 in. by 4 in. and one 4 in. by 5 in. Is it possible to do this and have some waste left? Why?

Set R4
Some interesting numbers (Fibonacci numbers)

a Can you extend this sequence to 17 terms? (Fibonacci's sequence.)

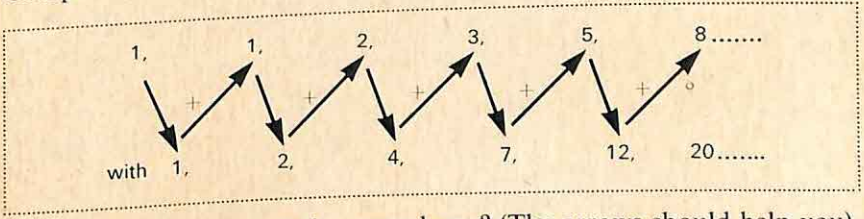
1, 1, 2, 3, 5, 8, 13, . . .

b Take any term of the series except the first, and multiply it by the next on the right. Now take the next to the right of that and multiply it by the one to the left of the first you chose. What is the difference between these numbers? Try it again with another number. Is the difference always the same?

c Take the eleventh term. Double it. Subtract from this result the twelfth term and compare this result with the ninth term. What do you find? Do a similar calculation with a term of your own choice. Does the same thing always work?

d If a term is called the *n*th term, what is the next term to the right called; and that two steps back to the left? Write down instructions for part c using these expressions in *n*.

e Compare the sequence



Can you see how we got the second one? (The arrows should help you). Extend it to sixteen terms. Now add one to each of the terms. What do you notice?

f What is the eleventh term of the Fibonacci sequence? What is its reciprocal? Work this out as a decimal fraction to six places. What do you notice? Can you make this carry on?

g Find any three successive numbers in the sequence. Work out the cubes of the numbers. Add the two greatest ones and subtract the smallest. Is the answer always a Fibonacci number?

h Work out the first and second differences for the sequence. What do you notice? Why does this happen?

i Find out who Fibonacci was and when he lived.

Set R5
Games with numbers

a The number 76125483 contains eight different digits. Which is missing? Multiply 76125483 by the missing digit. You get a nine-digit number. What do you notice about it?

b What do you notice about the digits of 1370258694? Divide it by 18. What is the result?

c In 1514 the German artist Albrecht Dürer engraved a picture called 'Melancholia'. In part of it he put this square of numbers which includes the year of the engraving:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

It is a 'magic square'.

Add up the numbers in the upper half of the square, and those in the lower half.
Add up the squares of these numbers in the same way.
Repeat this with the left side and right side of the square.
Add up each row, each column, and each of the two diagonals.
Add those in the four corners. Add the four in the centre.
Add the squares of the numbers in the first and third rows; add the squares of the second and fourth rows.
Do the same with the columns.
Add the squares of the numbers in each diagonal.
Repeat with cubes.
There is yet more. Can you find some facts we have not yet used?

d This is a pattern which goes on for ever all over the plane.

1	14	4	15	1	14	4	15	1
12	7	9	6	12	7	9	6	12
13	2	16	3	13	2	16	3	13
8	11	5	10	8	11	5	10	8
1	14	4	15	1	14	4	15	1
12	7	9	6	12	7	9	6	12

i How is it built up?

ii Select any row of four successive numbers and add them. Do this again with any other row. What do you notice? Is it really true? Try all the possible rows.

iii Repeat this with columns of four successive numbers.

iv Try again with four numbers taken from a line parallel to the sequence 1, 7, 16, 10.

This pattern is called a Nasik square.

e Here is another table for you to look at.

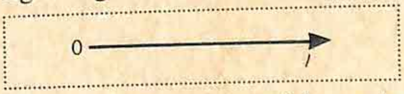
57	60	62	1	51	12	10	7
9	24	45	43	44	16	23	56
11	19	25	38	28	39	46	54
52	48	36	31	33	30	17	13
63	47	37	26	40	27	18	2
4	15	32	35	29	34	50	61
6	42	20	22	21	49	41	59
58	5	3	64	14	53	55	8

i Examine the inner two squares. Try the method of the previous question on them.

ii Examine the whole square. Is it a magic square like Dürer's? Is it as good as Dürer's?

Set R6
Another look at some of your geometry

a Look at the ray *below*. Imagine it rotating about the vertex 0. Through what angle(s) in degrees and in right angles must it rotate to form

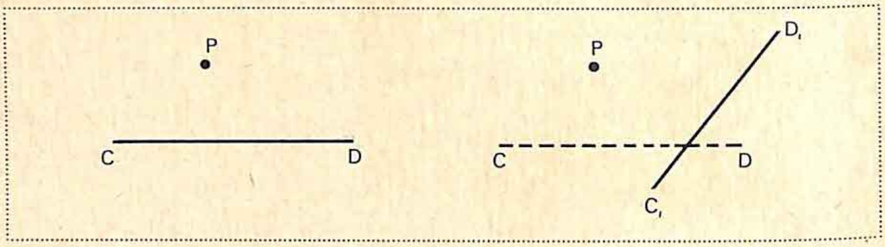


i a straight line?

ii a half-line?

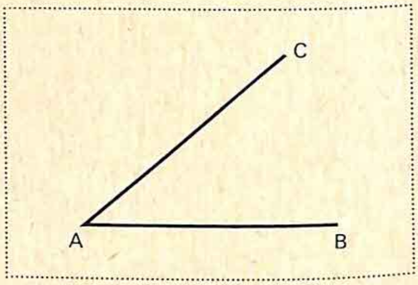
iii What will happen if it rotates through 720°, 1080°, 5 right angles, 540°?

b Suppose line segment CD rotates about the point P (not in the line set). It will take up position like this:



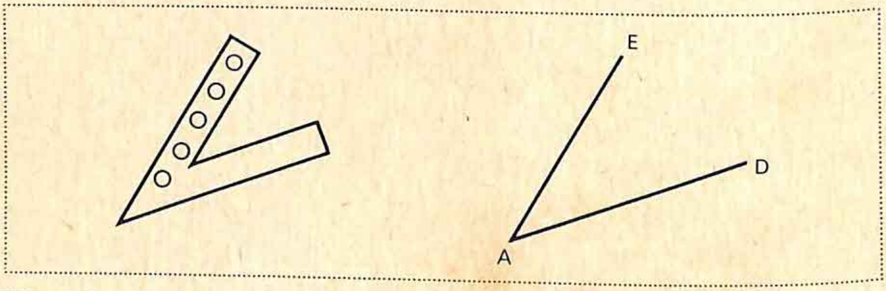
Draw a line from P at right angles to CD (such a line is the *perpendicular* from P to CD). Suppose it intersects CD in the point N. What path does N trace out as DC rotates? Through what angle will the line PN rotate to place C_1 D_1 at right angles to CD? and to make D_1 C_1 parallel to CD? and to make C_1 D_1 lie on top of (coincide with) CD?

c Imagine this figure rotating into a new position CA_1 B_1. About which point must it have rotated? (This point is the *centre of rotation.*)



Suppose it rotates about this point through 180°. What can be said of the segment AB and its new position B_1 A_1? What can be said of the angles CAB and CA_1 B_1? What sort of angles are they?

d Make a cardboard model of (c) by cutting two identical V shapes. Punch a number of holes along one arm of the angle. Hinge the two V's together with a split-pin and see what results you can get with various rotations and various centres of rotation.



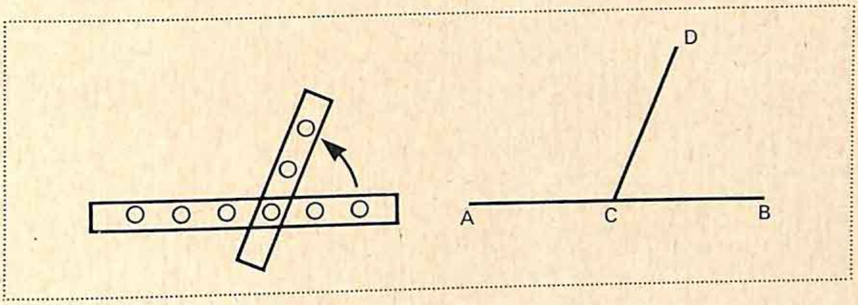
e Use point A as the centre of rotation. What amount of rotation will show a pair of vertically opposite angles? What do you know about vertically opposite angles?

f Could you have done all of these exercises equally well with an obtuse angled V?
Keep your V-shapes. They will be needed again.

Set R7
More geometry

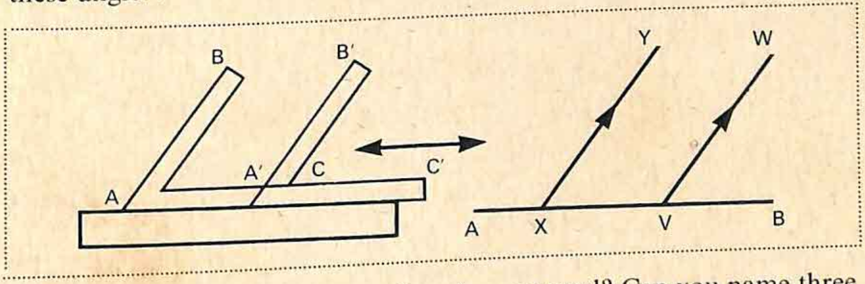
a Cut two strips of card $\frac{1}{2}$ in. wide and about 6 in. long. Punch holes at intervals along the middle of the strips. Pass a split pin through the two strips. Hold one strip still and rotate the other slowly. What sort of angles are formed? What do you know about them?

b Pass the split-pin through the end hole of one strip and a hole other than the end hole of the other (See the diagram). Rotate one strip as indicated and then answer the questions of part c.



c CD lies on CB and rotates anticlockwise. Through what angle will it rotate to lie on CA? If it has rotated through 90°, how much more has it to rotate to lie along CA? If it has rotated 60°, how much more? If it has rotated x°, how much more? What can you say about the measures of angles DCB and DCA *whatever position the line CD has reached*?

d Take your two V-strips and arrange them as in the diagram. Lay them against a ruler. Hold one still and slide the other in the direction of the arrows. (This sliding is called a *translation*). What can you say about the lines AB and A′B′? about the angles BAC and B′A′C′? What do we call these angles?



e Look at the figure. Which pairs of angles are equal? Can you name three pairs of angles which each add up to two right angles? What word do we use to describe this?

f Do you know a way of using the idea of (d) for drawing parallel lines? What do we use instead of V-shaped cards?

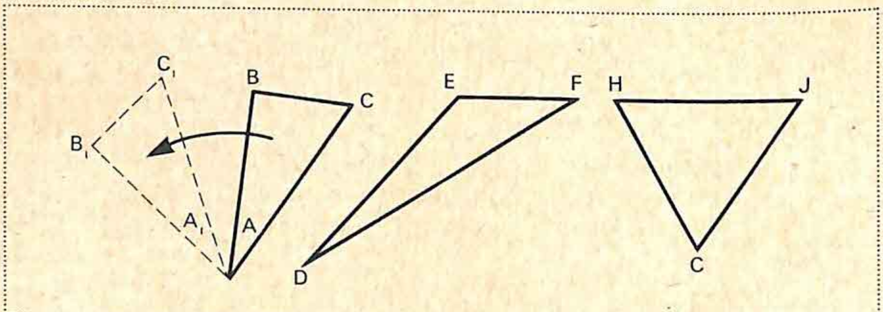
Set R8
Work with triangles

a Take two 60° set squares, exactly the same size. Arrange them in various ways so that two complete edges are touching and one set square is a 'reflection' of the other. What different figures can you obtain? Make sketches of the figures and mark the facts which you know about them from your experiment.

b Would the same results have been obtained with right angled triangles in which no angle was 60°? Would it be true when identical triangles, not right angled, were used? Try this with triangles cut from double thicknesses of paper.

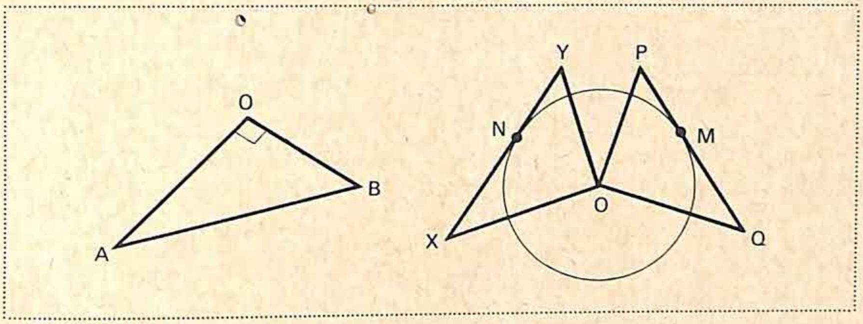
c What name do we give to identical triangles?

d Place your set squares on top of one another in the position ABC. Rotate one 180° in the direction of the arrow keeping A_1 touching A. Now translate $\triangle A_1 B_1 C_1$ so that A_1 reaches C and C_1 reaches A. What can you say about the sides BA and $A_1 B_1$ (two facts)? How many sides has the figure? What is the measure of the angle B? What can you say about BC and $C_1 B_1$? What sort of figure is it? What do you know about the diagonal of the figure?



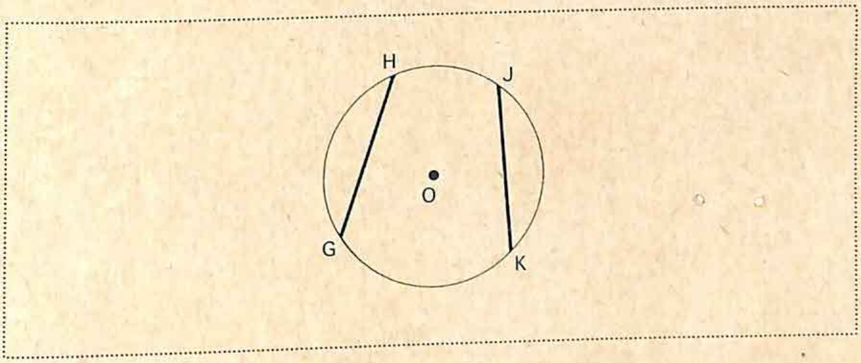
e Repeat the whole of Exercise d with a pair of triangles like $\triangle$DEF and another pair like $\triangle$GHJ. Cut your triangles out of cardboard.

f Cut a right angled isosceles triangle out of card. Rotate it through four right angles about 0; do this in four stages of 90°. Could similar results have been obtained from reflections? What figure has been traced out by this rotation in its four stages? What properties of this figure can you see from your experiment? What path was traced out by the mid-point of AB? What would be the differences if angle AOB were more than or less than 90°?



g Try to answer the following questions bearing in mind what you did in part (f). PQ and XY are lines touching a circle whose centre is at 0. (They are called *tangents*.) They touch the circle at M and N. What points of lines PQ and XY are M and N? If $X\hat{O}Y = P\hat{O}Q$, what do you know about XY and PQ? and about OX, OY, OP, OQ?

h In question (f) what paths were traced out by A and B? Now answer the following question. GH and JK are *chords* of a circle. In this case they are equal in length. What can you say about the distances of their mid points from 0? Euclid said 'Equal chords are equidistant from the centre of the same or equal circles'. What did he mean?



Set R9

1 {brown cats}, {bob-tailed cats} are subsets of {cats}.
Write down four other subsets.

2 {cars} has subsets A = {black cars}, B = {English cars}, C = {bubble cars}, D = {racing cars}.
Write descriptions of the following:

a $A \cap B$ **b** $A \cup B$ **c** $B \cap C$ **d** $C \cup D$ **e** $C \cap D$.

From what you know about cars, do you think any of these are empty sets?

3 If U = {mountains} there are subsets

A = {Mountains of the Moon}
B = {mountains more than 10,000 ft. high}
C = {Himalayan mountains}
D = {mountains in Scotland}
E = {Mount Everest}.

a Describe

i $B \cap C$ ii $B \cap D$ iii $C \cap E$ iv $C \cup E$ v $B \cap E$

b Is any of the subsets a subset of another one?

c Is {Ben Hur} a subset of D?

(You may use anything you know about mountains to answer this question.)

4 U = {detective novels} and some of the subsets are

A = {red covered detective novels}
B = {400 page detective novels}
C = {detective novels written in French}
D = {detective novels with more than 400 pages}
E = {detective novels with fewer than 400 pages}
F = {novels about Hercule Poirot}.

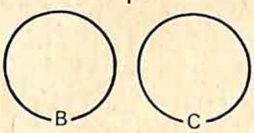
a Describe the following sets.

i $A \cup B$ iii $B \cap B$ v $D \cup E$ vii $C \cap F$
ii $A \cap B$ iv $B \cup D$ vi $B \cup D \cup E$

Is any of these equal to U? Are any of i–vii equal to each other? Which of i–vii do you think are likely to be empty sets?

b If we want to draw a picture of $B \cap C$ which, if any, of these should we draw?

I



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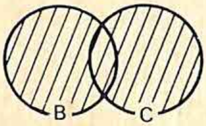

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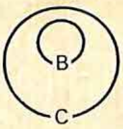
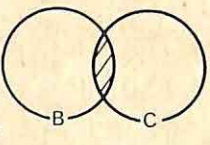
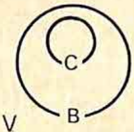




IV



V

c Would the statement 'Some French detective novels are bound in red' be equal to the statement '$A \cap C$ is not empty'? Explain your answer in words and with a diagram.

d Can you translate diagram b iii into words?

5 In this question you will not be able to use your knowledge of the sets.
U = {kelrads} A = {tacs} B = {ecims} C = {eloms}

a A, B and C are subsets of U. Write down statements to say this without using the *word* subset. Is there more than one way? Can you draw a diagram which shows that B is a subset of U?

b If $A \cap B$ is empty, what does this mean (in words)?

c Suppose $B \cap C$ is not empty, can you show this on a diagram?

d What does $B \cup C$ mean in words?

6 $U = \{0, 1, 2, 3, 4\}$
$A = \{1, 2, 3, 4\}$
$B = \{0, 1, 2\}$
$C = \{4, 3\}$
$D = \{0, 3, 4\}$
$E = \{1, 2, 5\}$
$F = \{\quad\}$

Write down the union and intersection of every pair of subsets of U in this list.

7 $U = \{5, 6, 7\}$
$A = \{5, 6\}$
$B = \{6, 7\}$
$C = \{6\}$
$\phi = \{\quad\}$
$A \cup B = \{5, 6, 7\} = U.$

Work your unions and intersection like this (in your head if you can). Fill in the following tables.

$\cup$	ϕ	U	A	B	C
ϕ					
U					
A					
B					
C					

$\cap$	ϕ	U	A	B	C
ϕ					
U					
A					
B					
C					

8 If $A = \{*, \S\}$ then $n(A) = 2$. $B = \{*, \S, \dagger\}$ and $C = \{\ddagger, \S, \pounds\}$.
What are $n(A \cup B)$, $n(A \cap B)$, $n(A \cup C)$, $n(A \cap C)$, $n(B \cap C)$?

9 If $D = \{1, 2, 3\}$ then $n(D) = 3$. Now $E = \{3, 4, 5\}$, $F = \{1, 2, 3, 4\}$, $G = \{0, 1, 2, 3, 4\}$

Which of the following statements are correct?

a $n(\mathrm{E}) = 5$

b $n(\mathrm{E}) = 3$

c $n(\mathrm{F}) = 1$

d $n(\mathrm{F}) = 4$

e $n(\mathrm{G}) = 4$

f $n(\mathrm{G}) = 0$

g $n(\mathrm{E} \cup \mathrm{G}) = 5$

h $n(\mathrm{G} \cup \mathrm{E}) = 6$

i $n(\mathrm{E} \cap \mathrm{F}) = 2$

10 Work out {multiples of 7}, {multiples of 9}, {multiples of 11} (put in 12 elements and fill up with dots).
What is the smallest number which 7, 9 and 11 will all divide? Are there other numbers into which they will all divide? If so, name one and say how you found it.

11

a Write down the set of numbers which leave a remainder of 2 when they are divided by 3. Call it C.

b Ditto – remainder 1. Call it B.

c What do you call those which leave no remainder? Call it A.

d What is $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$?

12 Write down:

a $\mathrm{A} = \{x \mid x > 6;\ n \text{ is a natural number}\}$

b $\mathrm{B} = \{y \mid y < 12;\ y \text{ is a natural number}\}$

c $\mathrm{C} = \{a \mid a < 15;\ a > 9;\ a \text{ is a natural number}\}$

d $\mathrm{D} = \{b \mid 1 < b < 7;\ b \text{ is a natural number}\}$

e $\mathrm{E} = \{e \mid 5 < e < 11;\ c \text{ is a natural number}\}$

f $\mathrm{F} = \{f \mid 6 < f < 8;\ f \text{ is a natural number}\}$

g $\mathrm{G} = \{x \mid 5 > x > 2;\ x \text{ is a natural number}\}$

h $\mathrm{H} = \{y \mid 3 < y;\ y \text{ is a mult. of } 2\}$

i $\mathrm{I} = \{x \mid 7 < x;\ x \text{ is a mult. of } 5\}$

j $\mathrm{J} = \{p \mid p < 12;\ p \text{ is a mult. of } 9\}$

k $\mathrm{K} = \{q \mid q < 10;\ q \text{ is a divisor of } 121\}$

l $\mathrm{L} = \{r \mid r < 0;\ r \text{ is a natural number}\}$

m $\mathrm{M} = \{s \mid 0 \leqslant s;\ s \text{ is a natural number}\}$

13 Using the sets of the last question, find the following:

a $A \cup B$

b $A \cap B$

c $B \cap C$

d $D \cap E$

e $E \cup F$

f $E \cup F \cup G$

g $K \cup L$

h $K \cap L$

i $L \cap M$

j $S \cap D$

k $S \cup F$

l $J \cup K$

m $H \cap I$

14 $U = \{a, b, c, d, e\}$

Make up a set of subsets of U so that the union and intersection of any two subsets is also one of the subsets. (For instance $A = \{d\}$ $\phi = \{\ \}$ is a solution because $U \cup A = U$, $U \cap A = A$, $U \cup \phi = U$ and so on. A set of *all* the possible subsets of U will do. Can you find other sets of subsets?)

15 Write down descriptions of the Universal sets which contain each of the following sets of subsets. Suggest two other subsets in each case.

a {tables}, {chairs}, {wardrobes}

b {lambs}, {ewes}

c {bottle}, {box}, {casket}

d {crabs}, {lobsters}, {shrimps}

e {comedies}, {tragedies}

f {epics}, {lyrics}, {ballads}

16 Say which of the following form partitions of a universal set and say what the set is in each example.

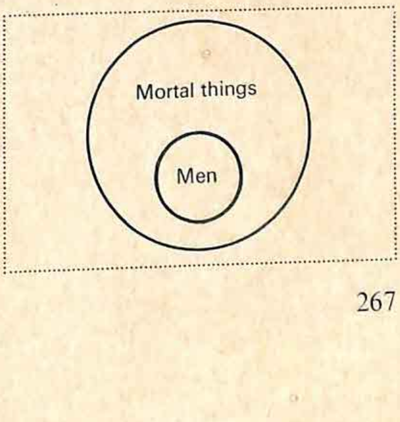
a {vowels}, {consonants}

b {even numbers}, {numbers which after division by 2 leave a remainder 1}

c {black cats}, {white cats}, {brown cats}

d {white horses}, {not-white horses}.

17 'All men are mortal' can be represented by this diagram:



a Are there, according to the statement or the diagram, any men who are not mortal things?

b Are there *necessarily*, according to the statement, any mortal things which are not men?

c If we wanted to draw a diagram to show that there were no mortal things apart from men, what sort of diagram would it be?

d Suppose some men were not mortal, then what would the diagram have to be?

e If no men were mortal, how would the diagram look?

18 Draw diagrams of the following. Read the statements to see what they mean but do not worry about whether they are true or not.

a All trees are green-leaved.

b All television programmes are good entertainment.

c All squares are rectangles.

d Some people are residents of Church Stretton.

e No residents of Church Stretton are not people.

f All residents of Church Stretton are people.

19 Read the following carefully to decide the meaning of the new symbol:
$\#(A) = 5$ if $A = \{a, e, i, o, u\}$
$\#(B) = 3$ if $B = \{b, \sharp, \natural\}$
$\#(C) = 6$ if $C = \{a, b, c, d, e, f\}$
Complete these statements:

a $\#(A \cup B) =$ **b** $\#(C \cup B)$ **c** $\#(A \cap C)$

What symbol have you used instead of # before?

20 If $A \subset B$ and $B \subset A$, what can you say about A and B?

Set R10 **1** Write down:

a The set all numbers which can be thrown with a die.

b The set of all couples (ordered number pairs) which can be thrown with two dice.

c The set of ordered triples which can be thrown with three dice.

d The set of numbers between 10 and 20 which are divisible by 3.

e $A\{x \mid 4 < x \leqslant 36;\ x \text{ is divisible by } 4;\ x \in N\}$

f $B\{x \mid 4 < x \leqslant 36;\ x \text{ is divisible by } 10;\ x \in N\}$

g The set of numbers between 4 and 37 which are divisible by both 4 and 10.

h $\{x \mid 10 \leqslant x \leqslant 20;\ x \text{ is even};\ x \text{ is not divisible by } 3;\ x \in N\}$

2 Give a description of each of the following sets and supply one more member of each set:

a {north, south, east, ...}

b {Matthew, Mark, Luke, ...}

c {London, Paris, Rome, Madrid, ...}

d {road, street, avenue, ...}

e {1956, 1960, 1964, ...}

f {1, 4, 9, 16, 25, ...}

g {1, 8, 27, 64, ...}

3 Find:

a $\{a \mid a+5 = 29\}$

b $\{b \mid b-7 = 15\}$

c $\{x \mid 3x = 39\}$

d $\left\{y \mid \frac{y}{5} = 6\right\}$

e From what system of numbers do all these values of variables come?

4 Solve the following equations, using whatever system of numbers you need to:

a $5x = 7$

b $7x = 5$

c $3y = 120$

d $4z+7 = 25$

e $9z-6 = 31$

f $6-9z = 5$

g $4(x+3) = 20$

h $\dfrac{x-4}{5} = 12$

Set R11 **1** **a** Draw a triangle XYZ in which YZ = 4·3 in., angle Y = 48°, and XZ = 3·2 in. Measure the angles *x* and *z*. Measure the side XY.

b Draw a triangle ABC in which BC = 7 cm., AC = 5·4 cm., AB = 2·8 cm. Measure the angles.

c Draw a triangle PQR in which QR = 2·6 in., angle Q = 53°, and angle R = 45°. Measure the angle P and the sides PQ and PR.

2 Can triangles be drawn with the following properties? Explain.

a Two obtuse and one acute angle.

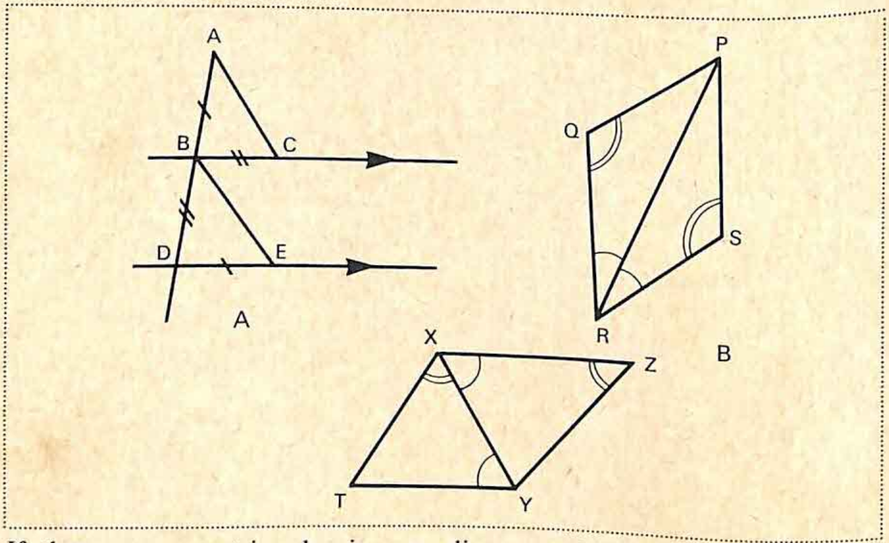
b One right angle and two acute angles.

c Sides of 2 units, 2 units and 5 units.

d Three obtuse angles.

3 **a** In which of these three diagrams is there a pair of congruent triangles?

b Say what sort of congruence it is and give the facts and reasons which make you sure.



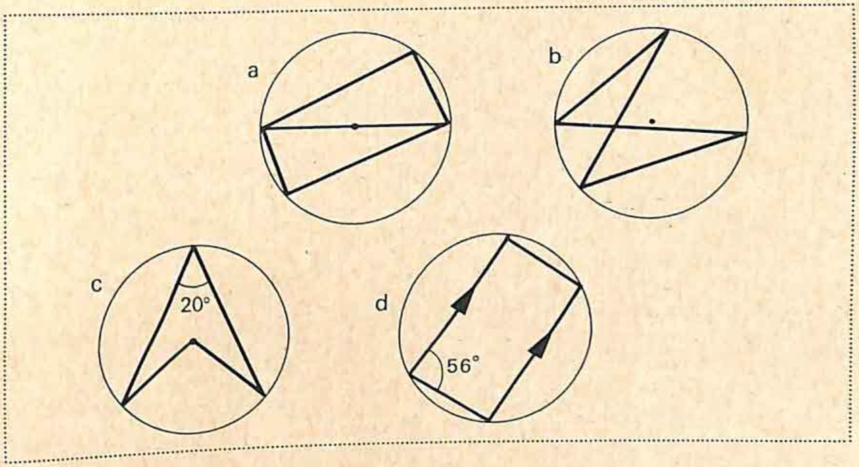
c If there are any triangles in any diagram which are not congruent, explain why they are not.

4 **a** What can you say about a polygon whose interior angles add up to four right angles?

b What can you say about a regular polygon whose interior angles add up to 900°.

c What can you say about a polygon whose exterior angles (formed by producing the sides in order, as below) add up to 360°?

5 In the diagrams:



a Mark the right angles.

b Mark the pairs of equal angles.

c Mark an angle of 40°.

d How big are the other angles in the figure?

Set R12 **1** **a** Write each of these following in the shortest possible way:

i $5y-10y+12y$

ii $a \times a \times a \times a \times b \times b$

iii $a^2 \times a^3$

iv $b^5 \div b^3$

v $3y^2+2y^2$

vi $\dfrac{a^6 \times a^7}{a^{12}}$

b Rewrite these expressions in a form without brackets:

i $7(y-4)$

ii $12(\frac{1}{2}y+1)$

iii $\frac{7}{9}(\frac{18}{7}-\frac{9}{7})$

iv $\dfrac{3x-6}{3}$

v $a \times (b \times c)$

vi $23(46+74)$

2 Solve the following equations:

a $\dfrac{3x+2}{2}=16$

b $\frac{3}{4}x+\frac{2}{3}=\frac{5}{6}$

3 What numbers could you put into the frames to make these into true statements?

a $\nabla + 5 + \nabla = 1 + \nabla + \square$

b $5^{\square} \div 5^2 = 5^4$

c $5^4 \div 5^4 = 5^{\triangle}$

4 Draw graphs of the following:

a $\{x \mid x < 7;\ x \in \mathrm{N}\}$

b $\{x \mid 2 < x < 5;\ x \in \mathrm{N}\}$

c $\{x \mid 12 \leqslant x < 15;\ x \in \mathrm{N}\}$

d $\{x \mid 9 > x \geqslant 7;\ x \in \mathrm{N}\}$

e $\{x \mid 7 \geqslant x > 2;\ x \in \mathrm{N}\}$

f $\{x \mid 8 \geqslant x \geqslant 6;\ x \in \mathrm{N}\}$

g $\{x \mid x > 6;\ x \in \mathrm{N}\}$

h $\{x \mid x < 5;\ x \in \mathrm{N}\}$

Set R13

1 Find numbers to fill the frames to make true statements of the following:

a $\square + 4 > 6$

b $3 + \nabla > 7$

c $4 + \triangle \geqslant 5$

d $\triangle - 1 > 4$

e $\triangle - 1 < 4$

f $\triangle - 1 = 4$

g $5 - \triangle < 8$

h $\square + 2 > 6$

i $\square - 2 < 6$

j $\square - 2 > 6$

k $7 < \triangle < 8$

l $7 \leqslant \triangle \leqslant 8$

m $7 \leqslant \square < 8$

2 List the following sets:

a $\left\{\frac{1}{y} \,\middle|\, 0 < y \leqslant 6;\ y \in \mathrm{N}\right\}$

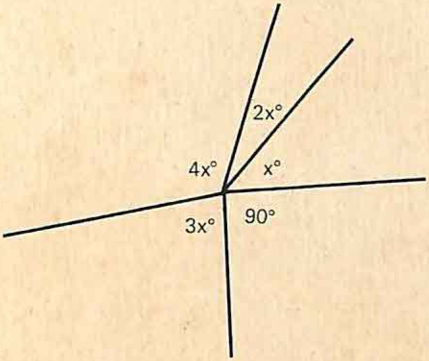
b $\left\{y = \frac{1}{z} \,\middle|\, 0 < z \leqslant 5;\ y \text{ is a rational number}\right\}$

c $\left\{y = \frac{a}{z} \,\middle|\, a < 4;\ z \leqslant 4;\ a, z \in \mathrm{N}\right\}$

Set R14

1 The minute hand of a clock has moved from 23 minutes past 2 to $14\frac{1}{2}$ minutes to 3. Through how many degrees has it turned?

2 Write down a statement as an equation in x and solve it to find x.



Symbols

$\cup$	union
$\cap$	intersection
$\{\}, \varnothing$	the empty set
$\in$	is an element of
$\subset$	is a subset of is contained in
$\supset$	contains
$n(A)$	the number of set A
$=$	equals
$>$	is greater than
$<$	is less than
$\mid$	such that
$\exists$	there exists
$\forall$	for all, for every
$\Leftrightarrow$	implies and is implied by is equivalent to
N	the set of natural numbers

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